An example of a musical palindrome: the minuet from Joseph Haydn's *Piano sonata No. 41, Hob. xvi/26*.

CHAPTER 6

The geometry of music

Wilfrid Hodges

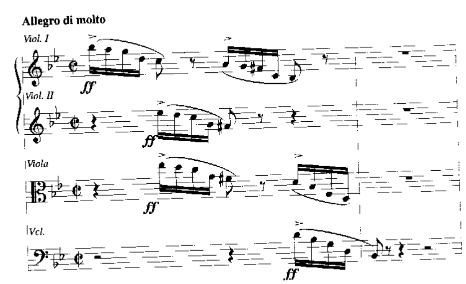
The dimensions of time and pitch make music into a two-dimensional space. Geometers study a space by describing its possible transformations, and they study a pattern in space by asking what transformations leave the pattern unchanged—that is, what symmetries the pattern has. We apply these ideas to musical space. For example, when does it make musical sense to squeeze a tune, or to turn it upside down? Since musicians cannot use very high or low pitches, a piece of music is like a frieze; we can find musical example of all the possible symmetries of a frieze pattern.

In memory of Graham Weetman (1963–92), mathematician and musician.

The rise and fall of musical space

While Edward Elgar was writing his *Enigma Variations*, he went for a walk along the banks of the River Wye with his friend G. R. Sinclair. Sinclair brought his bulldog Dan, who fell in the river and barked as he climbed out again. Sinclair turned to Elgar and said 'Set that to music'. So Elgar did, in the variation named *G.R.S.* after Sinclair. Elgar's manuscript marks 'Dan' at the point where Dan barks, though the printed editions rather prudishly leave it out.

Here is the score of the crucial moment:



Edward Elgar, *Enigma variations XI 'G.R.S.'*

You can clearly see the lines slanting down from top-left towards bottom-right. Since the music runs from left to right, these lines represent the music *falling*, and indeed you can hear it falling if you listen to a performance. Elgar makes the music fall because the dog fell. But these are two quite different kinds of falling. In the musical score, high-pitched notes (notes of short wavelength) appear near the top, and low-pitched notes (notes of long wavelength) are near the bottom. So a fall in the score indicates that the orchestra moves from short wavelengths to long wavelengths. The dog, on the other hand, falls by moving rapidly two or three metres closer to the centre of the earth.

Johann Jakob Froberger, *Suite XII in C major, Lamento sopra la dolorosa perdita della Real Mstà di Ferdinando IV, Ré de Romani.*

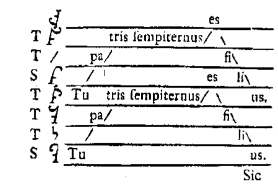
Here is an example of the opposite phenomenon: music that rises to describe something going up. It comes at the end of Froberger's musical depiction of the death of Emperor Ferdinand IV. The picture shows Froberger's manuscript, and you can see Froberger's picture of the clouds of heaven welcoming the soul of the emperor as it climbs up a scale of three octaves.

The two examples above illustrate the difference between up or down in space and up or down in musical pitch. In fact there are not two but three different kinds of space to be correlated. First, there is *physical space*. It has four dimensions—three of space and one of time. Second, there is *the score*. To a first approximation, the score is a plane surface with a horizontal dimension and a vertical dimension. By convention the horizontal dimension represents time, from the past on the left to the future on the right. Also by convention the vertical dimension represents pitch; notes of shorter wavelength are written higher up. Third, there is *musical space*. This space has any number of dimensions, depending on how we choose to analyse it. The two most obvious dimensions are time and pitch, and these are the two that we represent as dimensions in the score. Probably the best candidate for a third dimension is loudness. But the human ear is very bad at comparing the

loudness of different sounds, and even worse at remembering degrees of loudness. Most music doesn't distinguish more than five or six degrees of loudness.

Taken literally, time in music just is physical time; some musical events happen before others, and 'before' means 'earlier than' in the usual physical sense. It's pure convention that time is represented in the score by movement from left to right. The same convention makes physicists put time on the x-axis, moving forwards from left to right.

Pitch is a more complicated matter. To us today it seems obvious that high-pitched notes are 'high' and low-pitched ones are 'low'. That's how we are able to understand the music of Elgar and Froberger quoted above. So it comes as a shock to learn that in classical Greece high-pitched notes weren't heard as 'high'. In fact the highest-pitched note of the classical Greek octave was called *nete*, the 'nether' or lowest note. It got this name from the fact that the stringed instrument called a kithara was held with the highest-pitched string nearest the ground—just how one holds a guitar today. The classical Greek expression for high-pitched notes was *oxys* 'sharp', whence our musical sharps today; the Greek for low-pitched was *barys* 'heavy'. Classical Greek musicians represented pitches by letters, not by position on a musical page.



Late ninth-century notation, copies in print by Gerbert, *Scriptores ecclesiastici de musica*, 1784.

Very likely the correlation between short wavelength and height on the musical page was set up before anybody connected either of them with physical height. The correlation was made in western Europe, probably during the period 850–1150 AD. Two manuscripts of the late ninth century use a system of labelled boxes for the pitches, and they both put the boxes for higher pitches nearer the top of the page. This clumsy system never caught on. But during the next few centuries a notation developed for showing where the music rises and falls, and the notation formed a strong tendency to show rises in pitch by shapes like /, and falls in pitch by shapes like \. This led directly to the modern staff notation, which started to emerge in the twelfth century.

Nobody knows just when high pitch came to be correlated with physical height, but the correlation seems to come from western Europe again, and it became very strong during the fifteenth century. This is the century in which the names *altus* or *superius* 'high' and *bassus* 'deep' came to be used for high-pitched and low-pitched voices—whence our altos and basses. (Altos count as high because women weren't allowed to sing in churches.) During this century there were still a few notation systems that put lower-pitched sounds higher on the page—for example, some Italian lute tablatures—but they disappeared at the beginning of the sixteenth century, presumably because they had come to feel too unnatural.

Josquin Desprez put the seal on this development by sending Jesus down from Mount Olympus in a descending scale of twelve notes. This is still one of the longest descending motifs in all vocal music (see overleaf).

Josquin Desprez, *Huc me sydereo* (late 1490s).

Up, down, between and distance

Josquin's idea opened the floodgates to a torrent of musical representations of ups and downs, mostly in music from western Europe. We find them in Byrd, Purcell, Handel, Haydn, Wagner and Elgar.

An interesting example is a moving passage from the chorus *The cold deepens* in Michael Tippett's oratorio *A child of our time*. The first staff is the soprano part and the second staff shows some of the notes played by the orchestra.

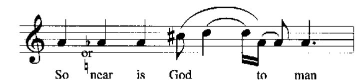
Michael Tippett, *A child of our time, No. 26 Chorus.*

While the soprano sings 'The world descends' and duly sinks downwards into the frozen ocean, the orchestral part surprises us by moving upwards. Tippett knows exactly what he is doing. Physical space has more structure in it than just up and down. It also has distance, and as distance changes in time we have moving apart and coming together. We can carry these notions over to musical space. In fact, Tippett's piece conveys a strong feeling that as the sopranos descend and the bass instruments of the orchestra rise to meet them, something is getting trapped between the two.

Franz Schubert, song: *Death and the maiden.*

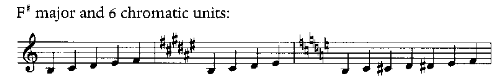
Tippett was by no means the first composer to play this metaphor. Schubert has a very similar device in his song setting of Claudius' poem *Death and the maiden*. In the first half of the song the maiden begs Death to leave her alone ('Rühre mich nicht an'). She sings energetically, but the bass line in the piano betrays that her strength is sinking. Then suddenly the bass line turns upwards; not because her strength comes back, but because Death is trapping her between the right hand and the left hand of the pianist. From that point onwards, only Death sings.

Charles Ives set himself an impossible problem. He wanted to use pitch distance to represent the fact that God is infinitely close to man. But what is an infinitesimally close pitch distance? In the end Ives gave up and left it to the singer to decide. Maybe what Ives wanted was a smallest perceptible pitch difference. There is no standard notation for this.



Charles Ives, *Duty.*

How does one measure distances of pitch? For classical western music there are two main answers. The first is that pieces of music tend to be in a key, and we count one unit of distance for each step up the scale of that key. This measure of distance is called the *diatonic metric*, and it depends on the key. ('Metric' is the mathematicians' name for a scale of measurement.) The second answer is that for many instruments with set pitches (such as pianos and organs) the smallest distance between two playable notes is a semitone; so we count one unit of distance for each semitone. This is the *chromatic metric*. Thus from B to F is 4 diatonic units in the scale of C major, 3 diatonic units in the scale of F♯ major and 6 chromatic units:

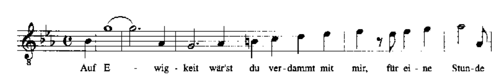


Metrics: diatonic in C major; diatonic in F♯ major; chromatic.

Things become immensely more complicated as soon as one moves even a short distance from western classical music.

There is a natural dual to Ives' question: How can we use musical space to represent that two things are *infinitely far apart*? For some reason, composers have generally wanted to do this more with time than with pitch; the problem is to represent eternity within the confines of a piece that lasts, say, half an hour. There are several ways to do it. A simple way is to make a note last not for ever but for a relatively long time. Thus Wagner in *Parsifal*:

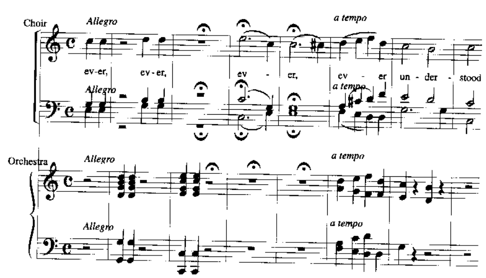
('You and I would be damned for eternity, for the sake of one hour.')



Richard Wagner, *Parsifal*, second act.

Here eternity lasts seven crotchets, compared with the two crotchets of an hour, giving the rather unimpressive ratio of 3.5 hours to 1 eternity.

Haydn in *The heavens are telling* from his *Creation* puts two pauses on 'ever', daring Gabriel (the soprano voice) to make them last as long as she can. (In the German version the pauses sit pointlessly on the word 'keiner'; but the libretto for *Creation* was written first in English.)



Joseph Haydn, *Creation*, chorus: *The heavens are telling*.

As we shall see below, other composers have used a subtler and more geometric way of pointing to eternity.

What is space?

Until the middle of the nineteenth century there was very little for mathematicians to say about musical space. This was because mathematicians had a shallow view of space itself; they thought of it as built up from points, lines, planes, and so forth. There is not much to be said about musical points and lines.

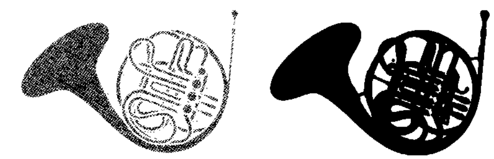
But the second half of the 19th century, particularly the work of Felix Klein, brought a new view of what space is. Instead of asking what space is made of, we ask what are the significant *transformations* of space. Roughly speaking, a transformation of space is a rearrangement of space and the things in it, that can be written by a simple mathematical formula.

Felix Klein (1849–1925)

Our musical space has just the two dimensions of pitch and time, so it forms a two-dimensional space—in fact, a plane. The figures below illustrate four kinds of transformation of a plane. The light horn is a set of points of the plane; the transformation moves the plane so that those points finish up forming the dark horn. The transformed version of a set of points in the plane is called the *image* of the set; so the dark horn is the image of the light one.

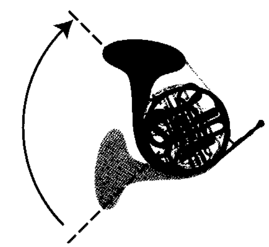
- A *translation* is a transformation T that moves all points of the plane in the same direction and through the same distance. (If the

distance is 0, then T is the *identity transformation* that leaves everything exactly as it was.)



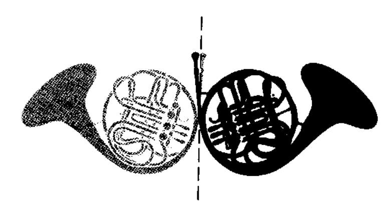
Translation

- A *rotation* is what the name suggests: it rotates the whole plane through some angle strictly between 0° and 360° around some fixed point. (In the illustration the fixed point is in the middle of the circle made by the horn's tubes, and the rotation is clockwise, as shown by the arrow.)



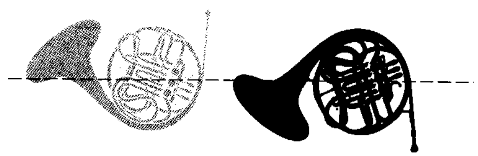
Rotation

- A *reflection* is what you get if you put a two-sided mirror at right angles to the plane and take each point to its reflection in the mirror. The dashed line shows the mirror.



Reflection

- A *glide reflection* is the hardest to describe. This transformation consists of a translation along a line (the dashed line of the illustration), followed by a reflection in the same line.



Glide reflection

These four kinds of transformation have an important property: they never change the distance between any two points on the plane. (This is sometimes expressed by saying that they move the plane 'rigidly'.) Transformations with this property are called *isometries* because they don't alter the scale of distances (the metric). There is a theorem of geometry which tells us that every isometry of the plane has one of the four types above.

We can apply these ideas to musical space as follows. At a first approximation, a musical motif M consists of a set of notes performed at certain pitches over certain time intervals; so it is a subset of musical space. A *symmetry* of M is an isometry of musical space that takes M back to M (although it may rearrange the points within M).

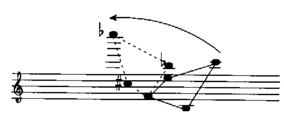


Figure from M. Kugel, 'Translation-Rotation', *Die Reihe* 7 (1965): the arrow indicates an isometry between the two quadrilaterals.

Since a piece of music lasts only a finite amount of time and uses only a finite range of pitches, a musical motif M lies within a bounded region of musical space; it has a start and a finish to left and right, and highest and lowest notes above and below. It follows that no translation (except the identity translation that keeps everything exactly where it was) can possibly be a symmetry of M. For example, if the translation moves points to the right, it will move the end of the motif to a point of time after the original motif is finished. By the same argument, a glide reflection can never be a symmetry of a musical motif. So only two kinds of symmetry are left: reflections and rotations.

There is no law of mathematics or music to stop composers from using any reflection or rotation they please as a symmetry of their motifs. But in practice, apart from some examples that are too trivial or too abstruse to be interesting, the symmetries of musical motifs turn out to be just two kinds of reflection and one kind of rotation, as follows:

reflection in a vertical line: call this transformation R_v
reflection in a horizontal line: call this R_h
rotation through 180° (exactly half a circle): call this R_2

These three kinds of symmetry are not independent; one can prove that if two of them are symmetries of a motif M, then the third is a symmetry of M too. So we can classify motifs according to which symmetries they have, and there are five possibilities, which we shall call the *symmetry types* of musical motifs. (The names *p1* etc. are adapted from crystallography, where one studies the shapes of crystals by describing their symmetries.)

p1: only the identity transformation is a symmetry;
ph: besides the identity, only R_h is a symmetry;
pv: besides the identity, only R_v is a symmetry;
p2: besides the identity, only R_2 is a symmetry;
phv: all of R_h, R_v and R_2 are symmetries.

For some broad guidance, here are examples of letters which (at least approximately) have the five symmetry types. We think of a letter as being a set of points of the plane. For example the reflection R_v is a symmetry of the letter A because the letter covers exactly the same set of points of the plane if you flip it over around a vertical line passing through the top of the A. On the other hand, if you turn A over around a horizontal line, the result is not A but an upside down A, covering a different set of points.

(p1)	F	G	J
(ph)	C	E	K
(pv)	A	M	V
(p2)	N	S	Z
(phv)	H	O	X

The five symmetry types of motifs.

Motifs of the five symmetry types

Type p1: no symmetries

The overwhelming majority of musical motifs belong here. For most of them, this is a fact of no particular significance. But suppose the composer has a mind to use some geometrical transformations such as reflections and rotations. Then an asymmetrical motif M gives much the best value, because its images under the different transformations are all different and distinguishable.

Given that most tunes are highly asymmetrical, what should we say about a composer who takes somebody else's tune, applies a transformation to it and then markets it as his own? This is exactly what Rachmaninov did to a violin caprice of Paganini. Below we show the Paganini original and the inverted Rachmaninov version in lock step. Rachmaninov follows Paganini bar by bar, and it's a chromatic inversion (so that it changes minor to major). But he slightly changes the rhythm and at one point he jumps by an octave. (It's a good exercise to play Paganini's tune upside down with no further alterations; one can see why Rachmaninov made the changes that he did.) What the table doesn't show is the difference in mood between a single violin playing staccato on its E string and a large symphony orchestra souping up the harmonies.

Upper line: Niccolo Paganini, *24 caprices for violin solo*, Op. 1, No. 24.
Lower line: Sergei Rachmaninov, *Rhapsody on a theme by Paganini*, Op. 43, *Variation XVIII*.

In the Classic FM Hall of Fame, where the British public votes for its most popular classical pieces, the scores in the year 2000 were

Rachmaninov inverted version: 33rd out of 300.
Paganini original: nowhere.

Type ph: only reversal of pitch

If a simple melody with no accompaniment has type *ph*, then it consists of a single note repeated. Surprisingly there are motifs with this property. Anton Reicha, a friend of Haydn, published a piano fugue whose subject consists of the same note repeated 34 times. (Towards the end the left hand sees the challenge and manages to repeat a single note 86 times. Fans of the Fibonacci numbers will be interested to hear that these 86 notes are grouped into blocks of 5 beats and 13 beats.)

Reicha's fugue is more entertaining than musical. On the other hand there certainly are worthwhile melodies that lie entirely in one pitch. But then they must owe their interest to another dimension. It could be rhythm, as with drum music, though a good drum player usually varies

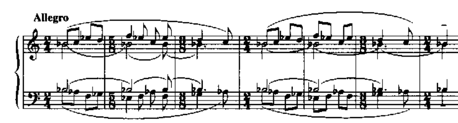
the timbre as well. The didgeridoo plays only one note, but an expert performer can get a tremendous range of timbres from it. Much of the music of the Italian composer Giacinto Scelsi revolves around changing the timbres of a small number of notes, as in his *Quattro pezzi su una nota sola*.

If a motif of type *ph* has several voices or instruments, the upper voices can reflect the pitch movements of the lower ones and there is no restriction to a constant pitch. When the upper and lower voices move in opposite directions, this is known as *contrary motion*. It forms a well-known gesture in classical music, just as in conversation one sometimes throws one's arms out or brings one's hands together. Some composers do it naturally, as if they never even noticed:

Wolfgang Mozart, *Clarinet quintet*, K381, opening of first movement.

Mozart's two upper voices reflect the movements of the two lower voices very closely for the first nine notes here. The intervals in the upper voices are not always exactly the same as in the corresponding lower ones, as they would be under a mathematical reflection; but they are remarkably close.

Most composers have little interest in making their upper voices mirror their lower ones with mathematical exactness, or in making the contrary motion last more than a few bars. But occasionally a composer does it for interest, as in Bartók's set of 'progressive piano pieces' for people learning to play.



Béla Bartók, *Mikrokosmos, No. 141, Subject and reflection.*

Incidentally, the title of Bartók's piece points to the geometric theme. But mirrors or reflections often appear in the titles of twentieth century compositions:

Boulez, *Constellation-Miroir* (in his 3rd Piano sonata)
Carter, *A mirror on which to dwell*
Debussy, *Reflets dans l'eau*
Francesconi, *String Quartet 3, 'Mirrors'*
Kokkonen, *... durch einem Spiegel ...*

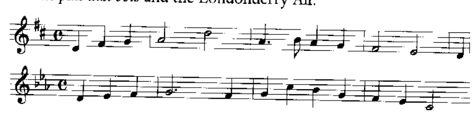
Maxwell Davies, *A mirror of whitening light*
Maxwell Davies, *Image, reflection, shadow*
Panufnik, *Reflections*
Ravel, *Miroirs*
Reynolds, *The behavior of mirrors*
Takemitsu, *Rocking mirror daybreak*
Certainly not all of these pieces contain pitch reflections.

Type pv: only reversal of time

One of the earliest recorded pieces of secular music has a strong leaning towards left-right symmetry. This is the jingle recorded on the wall of Reading Abbey, *Sumer is icumen in*.

Anonymous 13th century, *Sumer is icumen in*.

This tune is quite unusual in starting high, dropping and then rising again. A much commoner pattern in the folk music of western Europe (though not in Russia) is to start at a low pitch, rise to a high point and then fall back again. Two typical examples are the French folk song *Nous n'irons plus aux bois* and the Londonderry Air.

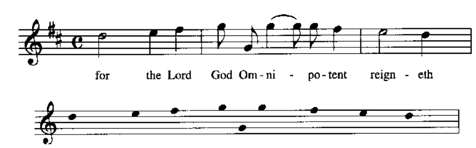


First line: French folk song: *Nous n'irons plus aux bois*.
Second line: *Londonderry air*.

The primitive rise-fall pattern is in some sense a grandfather of *sonata form*, where a section of rising tension, starting in the tonic key, is followed by a development section of high tension, and then by a final section where the tension falls and the tonic key is recovered. But in sonata form the final section is never a mirror image of the first section in any more precise sense. In fact musical palindromes, compositions with a virtually exact left-right symmetry, are fairly rare.

Some examples have a programme—these are usually vocal pieces with a libretto—and the symmetry expresses something in the programme. One very effective example of this genre is Stravinsky's depiction of Noah's flood spreading over the world and then receding, in his musical play *The flood*.

Another famous example, with a programme of a sort, has nothing directly to do with reflections and everything to do with balance and stability and the other virtues of a well-ordered state. Here is Handel implying, not quite subliminally, that The Lord God has everything very nicely under control, thank you.

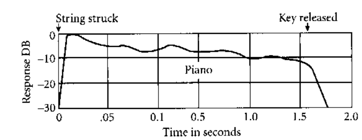


George F. Handel, *Messiah, Hallelujah chorus*.

There is a slight asymmetry in the rhythm, but not enough to damage the symbolism.

Some other musical palindromes seem to have been written for the challenge. If the symmetry is obvious enough, the performers can enjoy it as much as the composer. One of the best specimens of this kind is Haydn's reversible minuet. He was so proud of it that he recycled it into three separate works, the piano sonata shown at the beginning of the chapter, a violin sonata and a symphony.

In a sense, Haydn cheats with this piano sonata. If you take a tape recording of the first half of the minuet and play it backwards, you won't hear anything remotely like the second half. This is because of two physical properties of piano notes. First, they start with a bang and fade out gradually; so if you play them backwards, they start soft and finish with a bang, which makes it impossible to hear them as piano notes at all. And second, the length of time that they last depends on how loud they are (unless the pianist brings down the damper to stop them). This means that a loud note A may start before a soft note B and finish after B; when the tape is reversed, A still sounds as if it was played before B, so the order of the notes is not reversed as it is in Haydn's minuet.



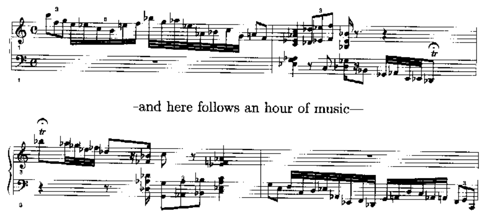
Piano note, from H. F. Olson, *Music, physics and engineering*, Dover, New York (1967), 257.

Luigi Nono's *Canti per tredici* is an exact palindrome for voices and not pianos, but it deliberately uses effects that are like what we hear when we reverse the tape of the piano recording. Like Alban Berg earlier in the twentieth century, Nono used palindrome as a structural device in composition. Most of Berg's and Nono's listeners will not notice these palindromes until they are pointed out, but for composer and performers they bind the music together as a unity.

Type p2: only rotational symmetry

This is not at all a common pattern, and generally it is not easy to hear. It hardly ever happens by accident, except where it falls out of some other feature of the motif.

As with the pattern *pv*, the non-accidental examples tend to be either technical challenges or programmatic symbols. A serious technical challenge should be a complete movement, or two together. This kind of extended symmetry only became possible in twentieth century music; the rules of earlier periods were too rigid. Hindemith provides an example in his piano piece *Ludus tonalis* ('game of tones'). If we ignore the very last chord, the final movement is the same as the first, but rotated through 180 degrees.



Paul Hindemith, *Ludus tonalis*, beginning and end.

Another example is Penderecki's orchestral *Threnody for the victims of Hiroshima*. This is one of many pieces which illustrate the fact that there is no contradiction at all between an emotionally charged topic and a highly formal compositional structure.

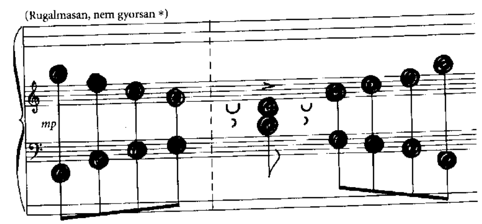
To illustrate the symbolic use of *p2*, here is an example with some interesting geometry that its composer may not have been fully aware of.

Nikolai Rimsky-Korsakov, theme from *The golden cockerel*.

The story of Rimsky-Korsakov's opera *The golden cockerel* revolves around a magic bird that sings two songs, one when there is danger and one when there is not. Rimsky-Korsakov has the ingenious idea of making the Safety song a geometric transformation of the Danger song. The tidiest way to do this is to choose a theme that has exactly two images under isometries; so it should be of type *ph*, *pv* or *p2*, not *phv* (which would make it identical under all isometries) or *p1* (which would give it four forms, not two). Should the song be flipped between Danger and Safety by a pitch reflection (as in *ph*) or a time reflection (as in *pv*)? By choosing *p2*, Rimsky-Korsakov gives the answer Yes to both possibilities.

Type phv: all possible symmetries

Interesting motifs of this type are extraordinarily rare. One appears in an elementary piano exercise of Georg Kurtág. The round blobs are instructions to hit the keyboard with the palm of your hand.



Georg Kurtág, *Játékok for piano I, Hommage à Eötvös Péter.*

Why are there so few examples of this type? I can only answer with an anecdote. Once I thought I heard an example in a concert of contemporary piano music. Since the composer (Luke Stoneham) was sitting behind me, I asked him in the interval whether I could check the score. When he heard what I was looking for, his jaw dropped and he said that if he had spotted any such figure in the piece, he would certainly have removed it. It seems that any composer with taste regards this symmetry group as too crass to use.

Breaking out of bounds

The dot-dot-dot symbol

•••

repays some study. We read it from left to right. The second dot comes from the first by a small translation to the right. If we repeat the translation, we get the third dot. That's enough to establish a pattern, and if we made a few more repetitions we would soon run over the edge of the paper.

So the three dots point us to infinity. This is a purely geometrical idea and it transfers immediately to the musical plane. Several composers have used it, usually at the end of a programmatic piece with a message 'and so life goes on'. One such composer is Bedřich Smetana, at the end of his string quartet *From my life* (see overleaf) where the story sinks into the indefinite future.

Béla Bartók used the same device at the end of his opera *Duke Bluebeard's castle*. Shortly before the final 'dot dot dot' we hear (in the German version) the word *ewig* 'ever' repeated four times. One can hear another example (played very softly) at the end of Benjamin Britten's opera *Peter Grimes* when Grimes is gone and the community's life returns to its normal cycle, while 'in ceaseless motion comes and goes the tide . . .'.

Did it have to be a horizontal translation that we used to point to infinity? Yes and no.

This calls for a small digression, to bring in a class of transformations of the plane that includes the isometries and more besides. Defined mathematically, *affine transformations* are the transformations which take any straight line to a straight line. One important kind of affine

Bedřich Smetana, end of string quartet: *From my life*.

transformation is *horizontal dilation* which keeps all pitches the same but slows down the time scale so that notes which were s seconds apart become rs seconds apart. The number r is called the *ratio* of the dilation. If $0 < r < 1$ then the dilation speeds up the time. Likewise a *vertical dilation* expands the pitch scale in some fixed ratio r but doesn't alter time.

Vertical dilations occur constantly in Beethoven's writing. He uses them systematically as a way of generating new material out of a basic motif. But the device is much older than Beethoven. There is a kind of canon called a *mensuration canon* where a tune is played simultaneously at two different speeds (and usually at different pitches too). This is a way of using horizontal dilations. It was popular in the fifteenth century, and in the twentieth century several composers used it, most notably Olivier Messiaen (as a metaphor) and Conlon Nancarrow (who used bizarre ratios like

$$\frac{1}{\sqrt[3]{\pi}} \Big/ \sqrt[3]{\frac{13}{16}} : \frac{1}{\sqrt{\pi}} \Big/ \sqrt{\frac{2}{3}}$$

in music for a player piano).

To come back to the matter in hand: there are just two kinds of affine transformation of the musical plane that can be iterated as often as we like but eventually lead out towards infinity. These are horizontal translations and horizontal glide reflections, either of which will give us the dot-dot-dot pattern. All other affine transformations of musical space that lead us out towards infinity hit the buffers after a very few iterations: either the pitch rises or falls too far for the instrument, or the music is too quick to be playable, or it's too slow to be heard as music, or some other similar physical problem.

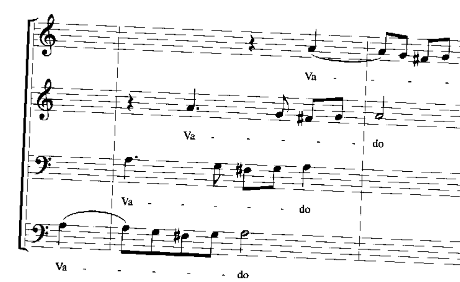
The best composers struggle against these limits, and where necessary they find ways of deceiving the ear into thinking there has been more iteration than in fact there has been. Two examples will suffice.

The first example is Handel fighting against the speed limits built into the action of an eighteenth century organ. The passage is from the organ part of his *Organ concerto in A major*.

George F. Handel, *Organ concerto in A major*.

We think he keeps doubling the speed of the repetition; this is a horizontal dilation with a ratio of 0.5. But when he reaches the physical limit, instead of continuing the iteration by repeating faster, he changes the notes. The ear is deceived. Handel may have learned this or a similar trick from the Italian opera writers.

The second is from one of those sadly beautiful motets that William Byrd wrote for his fellow Catholics (a persecuted minority under Elizabeth I) to sing at Ingatestone House under the protection of Lord Petre, *Non vos relinquam orphanos* 'I will not leave you comfortless'. Jesus is foretelling his ascension into heaven, *Vado* 'I am going'.



William Byrd, *Non vos relinquam orphanos*.

The moment passes quickly, but this was music to be appreciated by the performers themselves. The *Vado* motif seems to move steadily upward through the voices, pointing to Jesus' own movement upwards to heaven. Geometrically this is a diagonal translation iterated. In fact the movement is not as steady as it sounds; at two of the repetitions there is no movement upwards. Again the ear is deceived.

This passage of Byrd seems to have entered the subconscious of a number of later English choral composers. There is a very similar upward movement in a passage of Gustav Holst's *Hymn of Jesus* to the words 'When I am gone'; and Tippett has a splendid example in the climax of the final chorus of *A child of our time*, to the words 'Walk into heaven'.

Friezes

A *frieze pattern* is a pattern that repeats itself endlessly in one dimension. We can classify frieze patterns by their symmetries, just as we classified motifs. Since a frieze pattern keeps repeating, one of its symmetries must be a translation; this is one difference from motifs. Geometers looked to see what other isometries can be symmetries of a frieze pattern, and they discovered that there are exactly seven symmetry types of frieze. In the chart below, one should imagine each frieze pattern as running infinitely far to the left and the right. The names correspond to those used earlier, except that they also contain *t* for translation or *g* for glide reflection.

The seven types of frieze pattern.

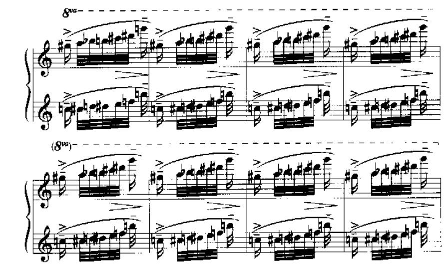
(*p1t*) FFFFFFFFFFFFFFFF
(*p1g*) FℲFℲFℲFℲFℲFℲFℲ
(*pht*) EEEEEEEEEEEEEEE
(*pvt*) AAAAAAAAAAAAAAA
(*pvg*) A∀A∀A∀A∀A∀A∀A∀
(*p2t*) NNNNNNNNNNNNNN
(*phvt*) HHHHHHHHHHHHHHH

A line of music endlessly repeated is a musical frieze pattern. One can find examples of all seven types. But one shouldn't expect too much here; music that repeats itself over and over again is almost by definition background or mood music, not meant to be listened to for its own sake. Nevertheless in at least four cases (*p1t*, *pvt*, *pvg* and *phvt*) there are interesting examples.

p1t

This is the type of a pattern that repeats over and over with no symmetries except sheer repetition. Many birds make sounds like this, from the *tap tap tap* of the woodpecker to the *jug jug jug* of the nightingale. Enrique Granados has a famous and suitably repetitive portrait of a nightingale at the end of *Quejas ó la Maja y el Ruiseñor* in his piano suite *Goyescas*. But obviously when birds are mentioned, we have to pay a visit to Olivier Messiaen. Here is slightly less than half of his setting of the song of the curlew. The rest is similar.

cious of a lar upward the words climax of heaven'. imension. classified tries must looked to , and they ze. In the infinitely ed earlier, ction. rn. One oo much y defini- r its own here are symme- from the tingale. ait of a no suite to pay a etting of



Olivier Messiaen, *Catalogue d'oiseaux, Le courlis cendré*.

pvt

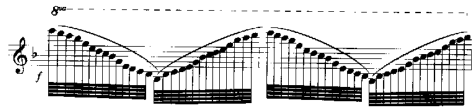
Earlier we saw that the symmetry type of an arch-shape that rises and then falls again is *pv*. So *pvt* is the type of a line of arches; we can see them in the row of As in the example above. Sibelius, that genius of orchestration, gives just such an arch shape to his violins to play over and over again. What makes this an interesting passage is that he does two other things. First, he divides the violins into four groups and makes each group start its arches at a different time. The effect is a throbbing sound that repeats at a quarter of the length of the arch; the arch lasts four bars but the combined pattern repeats at each bar.

Jean Sibelius, *Symphony No. 3*, last movement.

Second, although the violins are providing a background texture, the rest of the orchestra seems not to realise this and keeps trying to turn the rising side of the arch into a foreground tune. It never quite succeeds, but it holds us on the edge of our seats.

phg

A sine curve is the best mathematical example of this frieze type. What music sounds like a sine curve? There are plenty of rippling sounds in music, for example in Smetana's depiction of the Vltava in *Má Vlast*—though if you look at the score you will see that Smetana's ripples are generally a good deal less regular than the ear takes them to be. But step forward Debussy, whose jumping jacks leap to and fro across the sky in his *Fireworks* prelude. The symmetries are not quite exact, but with music like this, who's counting?



Claude Debussy, *Preludes for piano II, Feux d'artifice.*

phvt

This is the frieze type of a single note repeated regularly and endlessly. The endlessness and the resemblance to a church bell make this figure a potent symbol of death. A repeated note hangs over the last five pieces of Schubert's song cycle *Die schöne Müllerin*, sometimes dryly, sometimes frantically. In the song *Die liebe Farbe* it is constant throughout the piece. In the third bar below, Schubert (always a master of spacing) has placed a huge emptiness between the low $D^\sharp$ and the high $F^\sharp$ of the relentless bell. The fact that this is a major chord, which in romantic music tends to express happiness, makes the passage doubly poignant.

Franz Schubert, *Die schöne Müllerin, Die liebe Farbe.*

Three frieze patterns remain. In the following extract, which illustrates *p1g*, the upper staff is the cor anglais, while in the lower staff two bassoons alternately play the frieze motif the right way up and inverted. The inversion is chromatic.

Igor Stravinsky, *The rite of spring*, 14 in score.

The harp motif below illustrates *pht*; one can imagine a row of letter Cs opening up to the left instead of the right. The metric is diatonic in D minor.

Igor Stravinsky, *Petrushka*, 53 in score.

Finally, the flutes and oboes below play a motif that one can see as the crossbar of the repeated N for *p2g* in the table. The motif is made up of whole tones, so that again we are rotating in a chromatic metric.

Claude Debussy, *La mer*, second movement, bar 72.

Conclusion

Books on counterpoint or canon contain many examples of transformations of melodies. Beyond these, one must go to the composers themselves, the recordings and the scores. The following composers are particularly fertile in geometrical ideas:

Johann Sebastian Bach (1685–1750) was the grand master of fugue, and he wrote several collections of fugues which illustrate an amazing range of possibilities.

Béla Bartók (1881–1945) rivalled Beethoven in his ability to spin whole pieces of music out of a few notes by various geometrical transformations.

Ludwig van Beethoven (1770–1827) hardly needs introducing. C. Rosen, *The classical style*, Faber, London (1971), studies Beethoven's development of themes, and compares him in this regard with Haydn and Mozart.

Alban Berg (1885–1935) was, like Anton Webern, a student of Arnold Schoenberg. These three composers developed Schoenberg's twelve-tone techniques, which were built round isometric transformations of a sequence consisting of the twelve notes of a chromatic scale in some fixed order.

Josquin Desprez (c. 1440–1521) was one of a number of polyphonic composers in the period 1350–1500 who built much of their music around types of canon, sometimes of dazzling virtuosity. (Others were Machaut, Dunstable, Du Fay, Ockeghem.)

Joseph Haydn (1732–1809) loved musical tricks and witticisms, but he combined them with a deeply serious commitment.

Olivier Messiaen (1908–92) had an almost obsessive interest in structural devices—for example, scales with particular symmetry types, and rhythmic patterns from classical Indian music. Robert Sherlaw Johnson, *Messiaen*, J. M. Dent and Sons Ltd., London (1989), gives an excellent introduction to Messiaen's methods.

Conlon Nancarrow (1912–98) wrote almost exclusively for player pianos, because these instruments can produce notes with a speed and accuracy which no human player could possibly achieve. His *Studies for player piano* are a kind of modern *Art of fugue*, covering all conceivable kinds of canon. Unfortunately most of them are unpublished and exist only as piano rolls. But recordings have now been issued on CDs and are well worth hearing; the notes issued with the discs are a fascinating introduction.