148 RECHERCHES SUR LA NATURE ET LA PROPAGATION DU SON.

preuve de celui-ci tirée immédiatement de l'expérience; mais cet Auteur aura toujours le mérite d'avoir su en déduire avec une extrême simplicité la plupart des lois de l'harmonie, que plusieurs expériences détachées et aveugles avaient fait connaître.

Au reste, quelque principe qu'on adopte pour développer la nature des consonnances et des dissonances, il restera toujours à expliquer pourquoi il n'y a d'autres rapports primitifs consonnants que ceux qui sont contenus dans les nombres 1, 3, 5; car il est certain qu'une corde, qui sera la septième partie ou bien le septuple d'une autre, devra résonner dans le premier cas et frémir seulement dans le second, tout de même comme si elle rendait une douzième ou une dix-septième, d'où il résulte que, suivant même le principe de M. Rameau, on devrait regarder les rapports $\frac{4}{7}$ ou $\frac{7}{8}$ pour consonnants, ce qui est néanmoins démenti par l'expérience. Mais ce qui est plus étonnant, c'est que le rapport $\frac{8}{9}$, qui constitue une seconde majeure, est beaucoup moins dissonant que le rapport $\frac{7}{8}$, quoique les concurrences soient plus fréquentes dans celui-ci que dans l'autre. Il y a la même question à faire sur plusieurs accords qui ne sont pas reçus dans l'harmonie, quoiqu'ils contiennent moins de dissonances que d'autres qu'on emploie avec succès. Je crois que, dans quelque système de musique que l'on veuille imaginer, on ne pourra éluder ces difficultés qu'en recourant au goût et au sentiment commun, sur lesquels l'habitude et les préjugés ont peut-être beaucoup plus de pouvoir qu'on ne le pense ordinairement. Mais ce n'est pas ici le lieu d'entrer dans de telles discussions. Le savant M. d'Alembert en a traité fort au long dans l'article Fondamental de l'*Encyclopédie*, auquel nous nous contenterons de renvoyer.

Fig. 4.3. The conclusion of Lagrange's memoire of 1759 in Miscellana Taurinensia (OL, p. 148)

5 Musical Patterns

Wilfrid Hodges and Robin J. Wilson

In this article we look at several musical pieces that illustrate mathematical devices used by composers in their writing. These devices include canon, expansion, retrograde motion and inversion.

Canon

The first of these is *canon*, where the main theme is translated from voice to voice. Many canons are very familiar, such as *Three blind mice*, *Frère Jacques*,

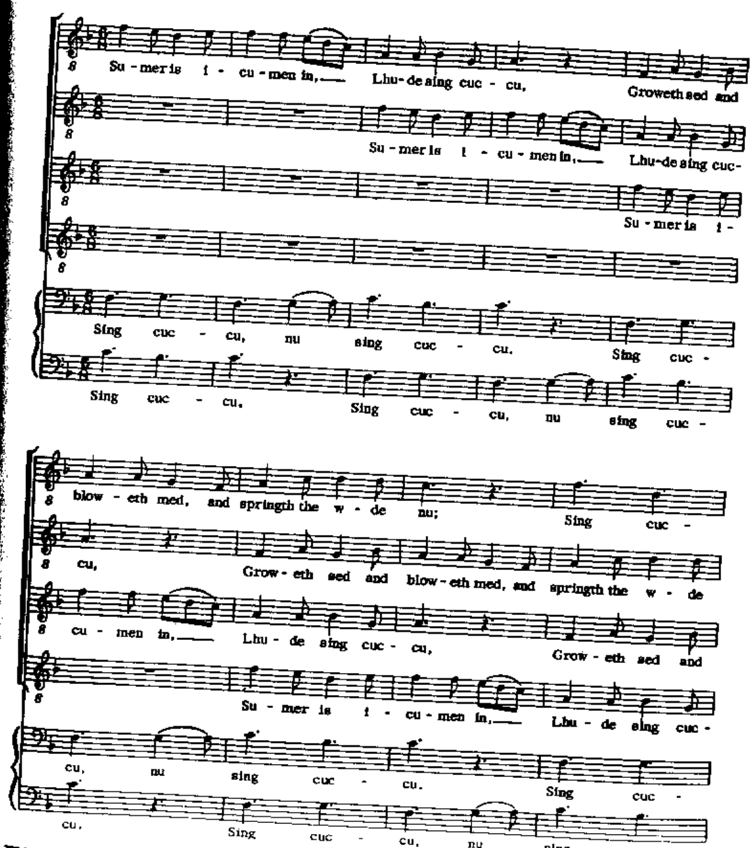

Fig. 5.1. Extract 1 – Sumer is icumen in

and *Row, row, row the boat.* One of the most remarkable canons ever written dates from about 1300 – *Sumer is icumen in.* Here the two lowest parts are in canon, and in addition the four high parts have their own canon above. The result when we put them all together: is shown in Fig 5.1.

In one of the most famous canons, the 16th-century *Tallis's canon*, the tenors sing exactly the same notes as the sopranos, but four beats later. This canon is shown in Fig 5.2.

Let's see what's happening mathematically. It's just the usual mathematical idea of translation, $y = x + c$. This is just one of several mathematical transformations that we can find in music – others include retrograde symmetry, where the music is the same backwards and forwards, and inversion, where we can turn it upside down, or combinations of these.

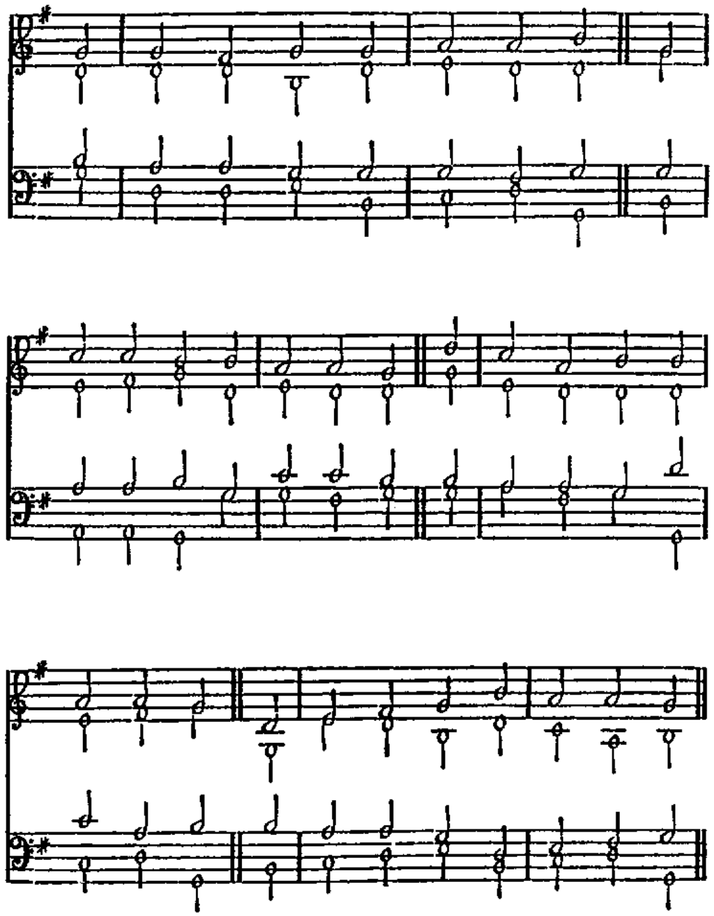

Fig. 5.2. Extract 2 – Tallis's canon

From the earliest times, composers seemed to enjoy setting themselves challenges, seeing whether they could compose acceptable music subject to given mathematical restrictions. There is even an unusual Tudor four-part canon in which three parts start in close canon, while the other part, a *cantus firmus*, sings a four-note phrase over and over again – but first with 8 beats on each note, then 7, then 6, and so on, down to 1 and then $\frac{1}{2}$.

Modified canon

Another challenge is to compose music in which the parts are in canon, but at different speeds – or $y = mx + c$, where $m \neq 1$. A brief but effective example of this occurs at an emotional high point of Brahms's *Requiem.* Here the sopranos sing 'I will see you again', which indeed we do with the tenors at half speed.

Fig. 5.3. Extract 3 – Brahms Requiem

There is a more complicated one, by Josquin des Prés from the 15th century – the *Agnus Dei* from his *Missa l'Homme Armé.* The bass line is also sung by the tenors, but at half speed starting on a different note, and by the sopranos in triple time. The final effect is complex and very striking. It is shown in Fig 5.4.

An even more complicated example is by the American composer Conlon Nancarrow, who died in 1998. In one of his canons, four voices play the same theme at speeds in the ratios 17:18:19:20, all entering at different times in such a way that most of the piece seems completely chaotic, but at the climax of the work they all come suddenly and miraculously into time with each other.

One of the most beautiful canons is the Chaconne from Henry Purcell's masque *Dioclesian.* Here the two recorder players are in exact canon throughout, the second just two bars after the first, while underlying this is a *ground*

Fig. 5.4. Extract 4 – Josquin Agnus Dei

bass – a six-bar phrase continually repeated throughout the piece. It is shown in Fig 5.5.

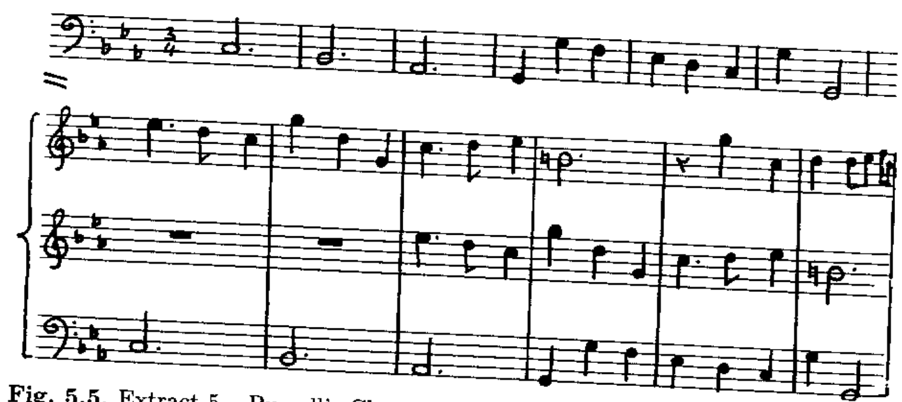

Fig. 5.5. Extract 5 – Purcell's Chaconne

Retrograde motion

Another mathematical device used by composers is that of retrograde motion. This yields pieces that can be played forwards or backwards, and are sometimes known as *palindromes*, or *crab canons* – here $y = -x$. Here is a palindrome from J.S. Bach's *Musical Offering* – reading the first line backwards gives the second line, and *vice versa*.

Fig. 5.6. Extract 6 – Bach's Crab canon

Another example is a delightful minuet by Joseph Haydn from his *Piano Sonata 41*, which the composer liked so much that he re-used it in his *47th Symphony*. Here's the beginning of it.

Fig. 5.7. Extract 7 – Haydn Piano Sonata 41

The earliest known piece of this type is by Guillaume de Machaut in the 14th century, who used this device in setting the words '*My end is my beginning*', and a recent example is by the American composer George Crumb, whose madrigal '*Why was I surrounded by mirrors*' is full of patterns that run the same way backwards as forwards. Such reflections were also used by Alban Berg in his opera *Wozzeck* – where large sections are the same backwards as forwards – and there's even a whole opera by Paul Hindemith which one can listen to either way round.

Inversion

The device of turning music upside down – or inversion – has been used effectively by several composers. The Hungarian composer Béla Bartók used it in his *Mikrokosmos*, in a piece called *Subject and reflection.*

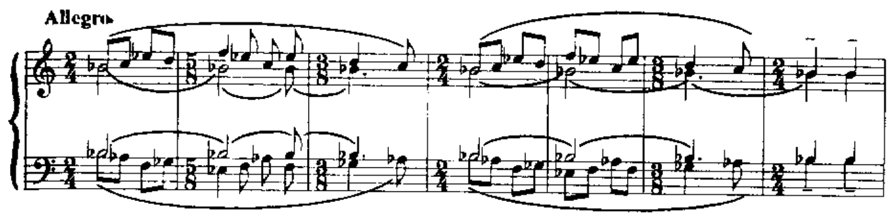

Fig. 5.8. Extract 8 – Bartók's Mikrokosmos

Several of J.S. Bach's compositions are full of mathematical symmetries. Here's an ingenious example of inversion: each of the following fugues is the inversion of the other.

Fig. 5.9. Extract 9 – two Bach fugues

An example that combines canon, reflection and inversion is Haydn's *Canon cancrizans*, where each of six sections is a three-part canon on the words, *Thy voice, O Harmony, is divine.* Each part in Section 2 is the same as one of the parts in Section 1 backwards, and the same thing happens also in later sections when the music is turned upside down! He wrote it on receiving an honorary degree from Oxford University in 1791.

Another interesting example of symmetry is Paul Hindemith's piano work *Ludus Tonalis*. Here the whole of the last movement is obtained by taking the first movement and playing it upside down and backwards. Apart from the last chord, rotating the whole movement through 180° does not change it at all.

—and here follows an hour of music—

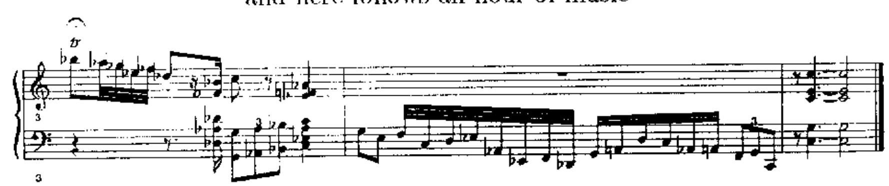

Fig. 5.10. Extract 10 – Ludus tonalis

The twelve-tone system

Many of these ideas are combined in twelve-tone music. In Schönberg's *Piano Suite* the twelve notes of the octave appear in some order, then transposed up six semitones, then inverted, and then the transposed version inverted, and they can also be played backwards. Figure 5.11 shows the result, with six forms of the tone row – the tones can appear in any octave.

Fig. 5.11. Extract 11 – Schönberg's Piano Suite