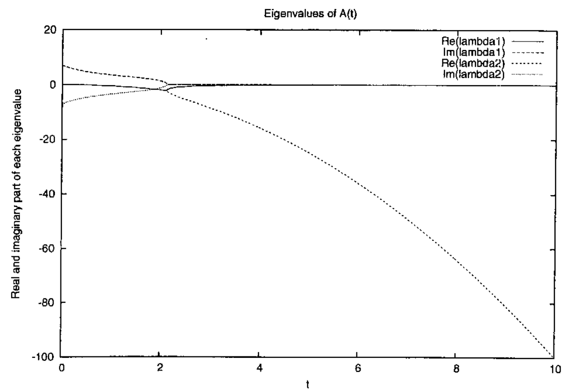
1. You solve the differential equation

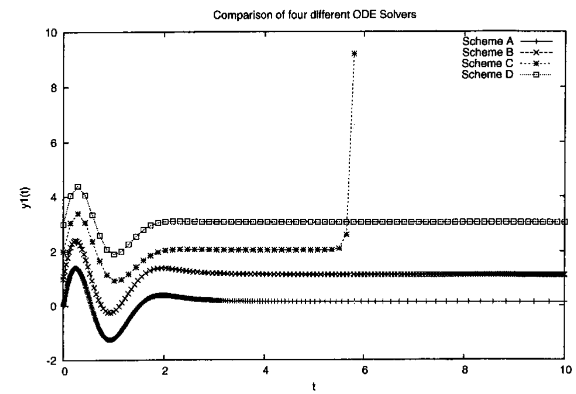
$$y'(t) = A(t)\,y(t)\,.$$

(a) Your choice of $A(t)$ is a time-dependent 2×2 matrix with eigenvalues as shown below. For which values of t do you consider the system "stiff"? Explain *briefly*!



- For $0 \leq t \lesssim 2$, the system is oscillatory (pair of complex-conjugate eigenvalues), with a slight damping as t increases.
- For $t \gg 2$, the system is stiff (one eigenvalue ≈ 0, the other $\ll 0$.)

(b) You try different solvers from your solver collection: RK4 and BDF4 with constant stepsize, and RK54 and Nordsieck-BDF4 with adaptive stepsize. Match the solvers to the solution graphs below, stating *briefly* one characteristic feature for each.



(Note that the curves are vertically shifted for better legibility, and only one of the components is shown.)

(5+10)

A: BDF 4 with adaptive step size
- large stepsize when the system is stiff
- small steps in the oscillatory region as order is lower than that of RK54, and error constants tend to be larger.

B: RK54 (is forced to take small steps as stiffness increase

C: RK 4 (scheme is unstable in stiff region → singularity)

D: BDF 4 with constant stepsize (not very accurate in oscillatory region, but stable)

2. For

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

find a plane rotation matrix R such that $R^T A R$ is diagonal. (10)

Solution 1: Write

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}}_{= \begin{pmatrix} 2c-s & 2s+c \\ c-2s & s+2c \end{pmatrix}}$$

$$= \begin{pmatrix} * & * \\ s(2c-s) + c(c-2s) & * \end{pmatrix}$$

$\Rightarrow$ Need $0 = s(2c-s) + c(c-2s)$
$= 2sc - s^2 + c^2 - 2sc$

$\Rightarrow$ $s^2 = c^2$ and also $s^2 + c^2 = 1$

$\Rightarrow$ $2s^2 = 1$

$\Rightarrow$ $s = \frac{1}{\sqrt{2}}$ and $c = \frac{1}{\sqrt{2}}$ (other choices of sign are possible)

$\Rightarrow$ $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Solution 2: Diagonalize A directly:

$$0 = \det(A - \lambda I)$$
$$= (2-\lambda)^2 - 1$$
$$= (1-\lambda)(3-\lambda)$$

$\Rightarrow$ The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$

Eigenvector for λ_1:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 \in \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

Eigenvector for λ_2:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v_2 = 0 \quad \Rightarrow \quad v_2 \in \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

Since A is symmetric, the eigenvectors are orthogonal, so by normalizing we obtain the orthogonal change of basis

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

3. Let A be a real symmetric $n \times n$ matrix, and set

$$R(x) = \frac{x^T A x}{x^T x}.$$

(a) If x is an eigenvector of A, show that $R(x)$ is the corresponding eigenvalue.
(b) From now on, let x be an arbitrary unit vector, and write

$$x = \sum_{i=1}^{n} \alpha_i v_i,$$

where v_i are the orthonormal eigenvectors of A with corresponding eigenvalues λ_i.
Show that

$$\lambda_{\min} \le R(x) \le \lambda_{\max},$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and the largest eigenvalue of A, respectively.
(c) Show that

$$\alpha_j = 1 - \frac{1}{2}\|x - v_j\|^2.$$

(d) **Extra credit:** Finally conclude that

$$R(x) = \lambda_k + O(\|x - v_k\|^2).$$

(5+5+5+*10*)

(a) If $Ax = \lambda x$, then $R(x) = \frac{x^T \lambda x}{x^T x} = \lambda$.

(b) $Ax = \sum_{i=1}^{n} \alpha_i \lambda_i v_i$

$x^T x = \sum_{i=1}^{n} \alpha_i^2 = 1, \quad x^T A x = \sum_{i=1}^{n} \lambda_i \alpha_i^2$

$\Rightarrow R(x) = \sum_i \lambda_i \alpha_i^2 \leq \sum_i \lambda_{max} \alpha_i^2 = \lambda_{max}$

$R(x) = \sum_i \lambda_i \alpha_i^2 \geq \sum_i \lambda_{min} \alpha_i^2 = \lambda_{min}$

(c) $\|x - v_j\|^2 = (x - v_j)^T (x - v_j)$

$= \underbrace{x^T x}_{=1} - 2 \underbrace{x^T v_j}_{=\alpha_j} + \underbrace{v_j^T v_j}_{=1}$

$= 2(1 - \alpha_j)$ since $x^T v_j = \sum_i \alpha_i \underbrace{v_i^T v_j}_{=\delta_{ij}}$

(d) $R(x) = \sum_{i=1}^{n} \alpha_i^2 \lambda_i$

$= \lambda_j \alpha_j^2 + \sum_{i \neq j} \lambda_i \alpha_i^2$

where $\alpha_j^2 = \left(1 - \frac{1}{2}\|x - v_j\|^2\right)^2$ by (c)

$= 1 + O(\|x - v_j\|^2)$

and $\left| \sum_{i \neq j} \lambda_i \alpha_i^2 \right| \leq \sum_{i \neq j} |\lambda_i| \alpha_i^2$

$\leq \max\{|\lambda_{min}|, |\lambda_{max}|\} \underbrace{\sum_{i \neq j} \alpha_i^2}$

$= 1 - \alpha_j^2$

$= O(\|x - v_j\|^2)$

$= O(\|x - v_j\|^2)$

$\Rightarrow R(x) = \lambda_j (1 + O(\|x - v_j\|^2)) + O(\|x - v_j\|^2)$

$= \lambda_j + O(\|x - v_j\|^2)$

4. (a) Let $Q = Q(t)$ be a time-dependent $n \times n$ matrix that satisfies the differential equation

$$Q' = QS, \qquad Q(0) = I, \qquad (*)$$

where $S(t)$ is a skew-symmetric $n \times n$ matrix, i.e. $S^T = -S$.
Show that $Q(t)$ is orthogonal for every $t \geq 0$.
Hint: You have to check that $Q^TQ = I$. Differentiate this relation and use the differential equation (*). or check that $QQ^T = I$

(b) **Extra credit:** Let A be a time dependent matrix that satisfies the so-called isospectral flow equation

$$A' = AS - SA, \qquad A(0) = A_0,$$

where S is skew symmetric.
Show that the eigenvalues of A remain unchanged under the evolution.
Hint: Use part (a) to conclude that $A(t) = Q^T(t) A_0 Q(t)$.

(10+*10*)

(a) $$\frac{d}{dt}(QQ^T) = Q'Q^T + QQ'^T = QSQ^T + Q(QS)^T = Q\underbrace{(S+S^T)}_{=0}Q^T = 0$$

Since $QQ^T = I$ at $t=0$, it will remain so for all $t \geq 0$.

(b) Define $Q(t)$ by (*). Then $S = Q^TQ'$

$$\Rightarrow \frac{d}{dt}(QAQ^T) = Q'AQ^T + \underbrace{QA'Q^T}_{} + QAQ'^T = 0$$

$$= Q(AS-SA)Q^T$$
$$= -Q(AS^T+SA)Q^T$$
$$= -QAQ'^T\underbrace{QQ^T}_{=I} - \underbrace{QQ^T}_{=I}Q'AQ^T$$

Since $QAQ^T = A_0$ initially, $A = Q^TA_0Q$ for all $t \geq 0$. I.e. $A(t)$ and A_0 are related through an orthogonal change of coordinates and, hence, have the same eigenvalues.

5. You solve the boundary value problem

$$-y''(x) = g(x)$$

on a non-uniform grid. I.e., the step size changes from one node to the next, and we define

$$h_k^+ = x_{k+1} - x_k,$$
$$h_k^- = x_k - x_{k-1}.$$

(a) Show that the local truncation error for the method

$$-2\left(\frac{y_{k-1}}{h_k^-(h_k^- + h_k^+)} - \frac{y_k}{h_k^- h_k^+} + \frac{y_{k+1}}{h_k^+(h_k^- + h_k^+)}\right) = g_k \qquad (*)$$

is given by

$$T_k = \frac{1}{3} y'''(x_k)(h_k^- - h_k^+) + O(h_k^-)^2 + O(h_k^+)^2.$$

(b) What is the (local) order of the method?

(c) **Extra credit:** Replace the right side of (*) by

$$\alpha g_{k-1} + (1-\alpha) g_{k+1}.$$

Determine α so that the local order of the method is at least 2.

(15+5+*10*)

(a) $$T_k = -2\left[\frac{y(x_{k-1})}{h_k^-(h_k^- + h_k^+)} - \frac{y(x_k)}{h_k^- h_k^+} + \frac{y(x_{k+1})}{h_k^+(h_k^- + h_k^+)}\right] - g(x_k)$$

$$= -2\left[\frac{1}{h_k^-(h_k^- + h_k^+)}\left(y(x_k) - y'(x_k)h_k^- + \frac{1}{2}y''(x_k)(h_k^-)^2 - \frac{1}{6}y'''(x_k)(h_k^-)^3 + \text{h.o.t.}\right)\right.$$

$$- \frac{y(x_k)}{h_k^- h_k^+}$$

$$+ \frac{1}{h_k^+(h_k^- + h_k^+)}\left(y(x_k) + y'(x_k)h_k^+ + \frac{1}{2}y''(x_k)(h_k^+)^2 + \frac{1}{6}y'''(x_k)(h_k^+)^3 + \right.$$

$$- g(x_k)$$

$$= y(x_k)\left[-2\left(\frac{1}{h_k^-(h_k^-+h_k^+)} - \frac{1}{h_k^- h_k^+} + \frac{1}{h_k^+(h_k^-+h_k^+)}\right)\right]$$

$$+ y'(x_k)\left[-2\left(\frac{-h_k^-}{h_k^-(h_k^-+h_k^+)} + \frac{h_k^+}{h_k^+(h_k^-+h_k^+)}\right)\right]$$

$$+ y''(x_k)\left[-\frac{(h_k^-)^2}{h_k^-(h_k^-+h_k^+)} - \frac{(h_k^+)^2}{h_k^+(h_k^-+h_k^+)}\right]$$

$$+ y'''(x_k)\left[\frac{1}{3}\left(\frac{(h_k^-)^3}{h_k^-(h_k^-+h_k^+)} - \frac{(h_k^+)^3}{h_k^+(h_k^-+h_k^+)}\right)\right]$$

$$+ \text{h.o.t.} - g(x_k)$$

$$= -2\,y(x_k)\,\frac{h_k^+ - (h_k^- + h_k^+) + h_k^-}{h_k^- h_k^+ (h_k^- + h_k^+)}$$

$$+ 2\,y'(x_k)\left[\frac{1}{h_k^+ + h_k^-} - \frac{1}{h_k^+ + h_k^-}\right]$$

$$- y''(x_k)\,\frac{h_k^- + h_k^+}{h_k^- + h_k^+}$$

$$+ \frac{1}{3} y'''(x_k)\,\frac{(h_k^-)^2 - (h_k^+)^2}{h_k^- + h_k^+} + \text{h.o.t.} - g(x_k)$$

$$= \underbrace{-y''(x_k) - g(x_k)}_{\equiv 0} + \frac{1}{3} y'''(x_k)\frac{(h_k^- + h_k^+)(h_k^- - h_k^+)}{h_k^- + h_k^+} + h.$$

$$= \frac{1}{3} y'''(x_k)\,(h_k^- - h_k^+) + \text{h.o.t.}$$

(b) To be able to speak about order, we need to introduce a stepsize parameter

$$h = \max_k \{h_k^+\} \qquad \left(= \max_k \{h_k^-\}\right)$$

From (a) it is clear that if $h_k^- \neq h_k^+$, then

$$|T_k| \leq \frac{1}{3}\,|y'''(x_k)|\;h + O(h^2),$$

i.e. the method is of order one. If $h_k^- = h_k^+$, the (local) order is at least two.

(c) $\alpha\, g(x_{k-1}) + (1-\alpha)\, g(x_{k+1})$

$$= \alpha\left(g(x_k) - g'(x_k)\,h_k^- + O(h^2)\right)$$
$$+ (1-\alpha)\left(g(x_k) + g'(x_k)\,h_k^+ + O(h^2)\right)$$
$$= g(x_k) + g'(x_k)\left[-\alpha h_k^- + (1-\alpha)\,h_k^+\right] + O(h^2)$$

$$\Rightarrow T_k = \overset{=0}{-y''(x_k) - g(x_k)} + \frac{1}{3} y'''(x_k)\,(h_k^- - h_k^+) + g'(x_k)\left(-\alpha h_k^- + (1-\alpha)\,h_k^+\right) + O(h^2)$$

The $O(h)$-term vanishes if $\frac{1}{3}(h_k^- - h_k^+) = -\alpha h_k^- + (1-\alpha)\,h_k^+$

$$\Rightarrow \boxed{\alpha = \frac{h_k^+ - \frac{1}{3}(h_k^- - h_k^+)}{h_k^- + h_k^+} = \frac{1}{2} + \frac{5}{6}\,\frac{h_k^+ - h_k^-}{h_k^+ + h_k^-}}$$