HW 7 - Solutions

1. $p_\alpha(x) = f(x) + \frac{\alpha}{2} \|h(x)\|^2$

let x^* denote the minimizer of p_α,

y^* denote the minimizer of p_β,

$0 < \alpha < \beta$

(a) $p_\alpha(x^*) \leq p_\alpha(y^*)$ since $p_\alpha(x)$ is minimized by $x = x^*$

$\leq p_\beta(y^*)$ by definition of p_α

(b) $p_\alpha(x^*) \leq p_\alpha(y^*)$ since $p_\alpha(x)$ is minimized by $x = x^*$

$p_\beta(y^*) \leq p_\beta(x^*)$ since $p_\beta(x)$ is minimized by $x = y^*$

$$\Rightarrow p_\alpha(x^*) + p_\beta(y^*) \leq p_\alpha(y^*) + p_\beta(x^*)$$

$$\Rightarrow f(x^*) + \frac{\alpha}{2}\|h(x^*)\|^2 + f(y^*) + \frac{\beta}{2}\|h(y^*)\|^2 \leq f(y^*) + \frac{\alpha}{2}\|h(y^*)\|^2 + f(x^*) + \frac{\beta}{2}\|h(x^*)\|^2$$

$$\Rightarrow (\alpha-\beta)\|h(x^*)\|^2 + (\beta-\alpha)\|h(y^*)\|^2 \leq 0$$

$$\Rightarrow \underbrace{(\beta-\alpha)}_{>0}\left(\|h(y^*)\|^2 - \|h(x^*)\|^2\right) \leq 0$$

$$\Rightarrow \|h(x^*)\| \geq \|h(y^*)\|$$

(c) $p_\alpha(x^*) \leq p_\alpha(y^*)$

$$\Rightarrow f(x^*) + \frac{\alpha}{2}\|h(x^*)\|^2 \leq f(y^*) + \frac{\alpha}{2}\underbrace{\|h(y^*)\|^2}_{\overset{(b)}{\leq} \|h(x^*)\|^2}$$

$$\Rightarrow f(x^*) \leq f(y^*)$$

2. $\nabla p_\alpha = \nabla f + \alpha h^T \nabla h$

$$\Rightarrow \text{Hess } p_\alpha = \text{Hess } f + \alpha \sum_{i=1}^{q} h_i \text{ Hess } h_i + \alpha (\nabla h)^T \nabla h$$

Let x^* solve the constrained problem, so that, in particular,

$h(x^*) = 0 \Rightarrow \text{Hess } p_\alpha(x^*) = \text{Hess } f(x^*) + \alpha \nabla h(x^*)^T \nabla h(x^*)$

Since $q < n$, $(\nabla h)^T \nabla h$ is singular, so that we can find two unit vectors

$u \in \text{Ker } (\nabla h(x^*))^T \nabla h(x^*)$

$v \perp \text{Ker } (\nabla h(x^*))^T \nabla h(x^*)$.

We find

$u^T \text{Hess } p_\alpha(x^*) u = u^T \text{Hess } f(x^*) u = \text{const}$

$$v^T \text{Hess } p_\alpha(x^*) v = v^T \text{Hess } f(x^*) v + \alpha \underbrace{v^T \nabla h(x^*)^T \nabla h\, v}_{>0}$$

$\longrightarrow \infty$ as $\alpha \to \infty$.

$\Rightarrow$ as $\alpha \to \infty$, $\lambda_{max}(\alpha)/\lambda_{min}(\alpha) \to \infty$.

3. $$\nabla F = \begin{pmatrix} \mathrm{Hess}\,(f + h^T\lambda) & \nabla h^T \\ \nabla h & -\mu I \end{pmatrix}$$

We want to show that this matrix is invertible at $x = x^*$, $\lambda = \lambda^*$, $\mu = \mu^* = 0$. Thus, we have to show that

$$\nabla F\,(x^*, \lambda^*; \mu^*)\, v = 0$$

has only the trivial solution $v = 0$.

Write $v = (u, w)^T$, so that

$$\mathrm{Hess}\,(f + h^T\lambda^*)(x^*)\, u \;+\; \nabla h(x^*)^T w = 0 \qquad \text{(A)}$$

$$\nabla h(x^*)\, u = 0 \qquad \text{(B)}$$

Multiply (A) with u^T from the left. Due to (B), $u^T\, \nabla h(x^*)^T\, v = 0$, and

$$u^T\, \mathrm{Hess}\,(f + h^T\lambda^*)(x^*)\, u = 0 .$$

Since the sufficient condition is assumed to be satisfied, this can only be true if $u = 0$.

Moreover, since x^* is a regular point,

$$\nabla h\,(x^*)^T\, w = 0$$

implies that $w = 0$. Thus, ∇F is a regular matrix.