Numerical Methods II

Problem Set 3

due in class, March 3, 2004

1. (From SM, p. 103.) Consider a matrix of the form

$$A = \begin{pmatrix} b_1 & c_1 & & & \cdots & 0 \\ a_2 & b_2 & c_2 & & & \vdots \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & & & a_n & b_n \end{pmatrix}.$$

   (a) Show that if
   $$|b_j| \geq |a_j| + |c_j| \qquad (*)$$
   for $j = 1, \ldots, n$ (with the convention that $a_1 = c_n = 0$), and
   $$|c_j| > 0 \qquad (**)$$
   for $j = 1, \ldots, n-1$, then the LU factorization of A can be computed without pivoting.

   Note: For simplicity, you may assume that all components of A are nonnegative, though the statement is true with arbitrary signs.

   Hint: See last semester's final exam problem (3b); solutions are online now.

   (b) Show that if (*) is satisfied with strict inequality in at least one component, then A is nonsingular.

   (c) Give an example where (*) holds, but (**) is violated, and where LU decomposition without pivoting is not possible.

   (d) Explain why the result from (b) is important for solving linear second order boundary value problems using second order finite differences (as, for example, in Project 4 below).

2. (From SM, p. 381) The differential equation

$$-y'' + r(x)\, y = f(x)$$

is approximated by

$$-\frac{\delta^2 y_j}{h^2} + \alpha\, r_{j-1}\, y_{j-1} + (1 - 2\alpha)\, r_j\, y_j + \alpha\, r_{j+1}\, y_{j+1} = \alpha\, f_{j-1} + (1 - 2\alpha)\, f_j + \alpha\, f_{j+1}\,.$$

   (a) Show that the local truncation error is at least $O(h^2)$ for any choice of α.

   (b) Find a particular value for α so that the local truncation error is $O(h^4)$.

3. Consider the equation

$$y'' + a^2\, y = 0\,.$$

   (a) Show that, when the boundary conditions are homogenous, i.e. $y(0) = y(1) = 0$, the equation has multiple solutions for special values of a.

   (b) Show that with inhomogenous boundary conditions $y(0) = 0$ and $y(1) = 1$, the equation has no solution for those values of a.

4. **Project:** Solve the boundary value problem

$$\begin{aligned} -y'' + \cos^2(x)\, y &= x\,, \\ y(0) = y(10) &= 0\,, \end{aligned}$$

using the standard second order finite difference approximation

$$-\frac{\delta^2 y_j}{h^2} + \cos^2(x_j)\, y_j = x_j\,.$$

Verify that the method is order 2 on a log-log error plot.

5. **Project:** Modify your code from the previous project to using the improved method from question (2). Verify that this method is order 4 on a log-log error plot.

6. **Project:** Solve the same boundary value problem using simple shooting.