Numerical Methods II

Final Exam

May 21, 2004

1. You use three different iterative methods to find the minimum of the function

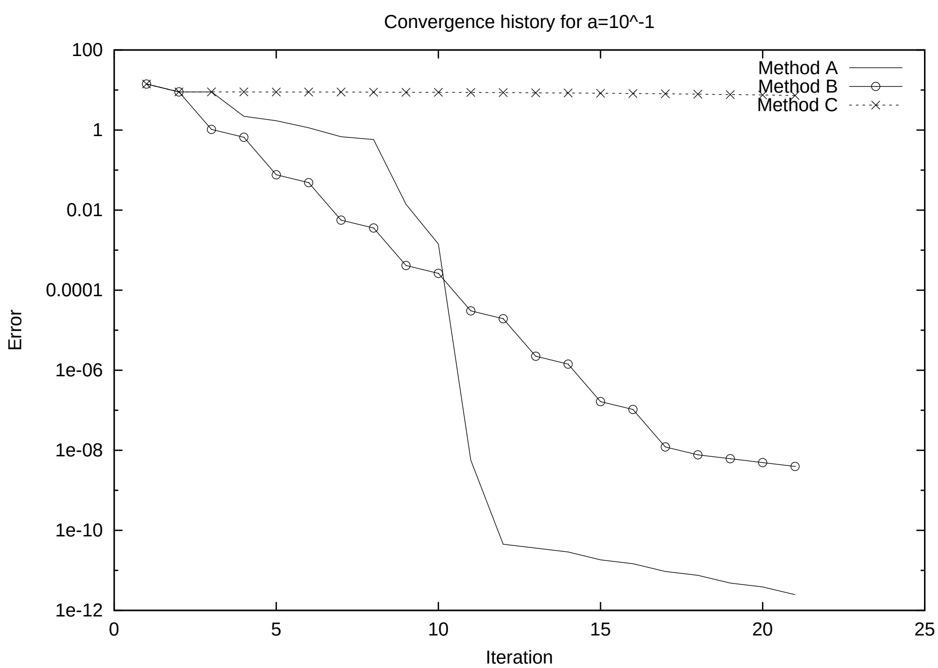
$$f(x, y) = -\frac{1}{1 + x^2 + a\,y^2}\,,$$

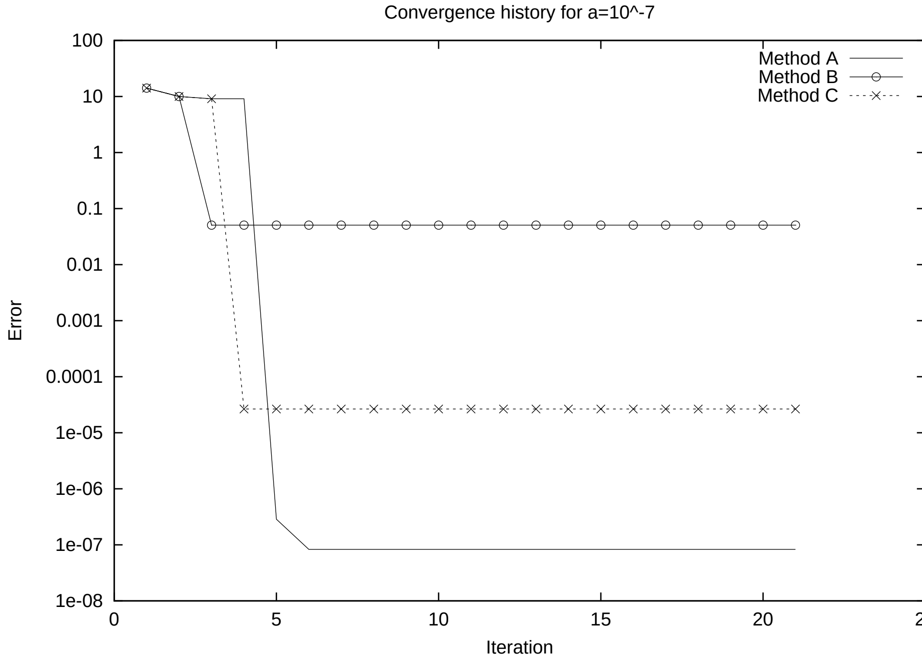
   (a) The gradient method;

   (b) The Fletcher–Reeves conjugate gradient method without restart;

   (c) The Fletcher–Reeves conjugate gradient method with restart every second iteration.

   The following two graphs show the decrease of the error with the number of iterations for two different values of a.





Match the method used to the graphs shown (the labeling is the same in both plots!), and explain your choice. (10)

2. Show that the stochastic differential equation

$$\mathrm{d}X = \tfrac{1}{3}\, X^{1/3}\,\mathrm{d}t + X^{2/3}\,\mathrm{d}W\,,$$
$$X(0) = X_0\,,$$

is solved by

$$X(t) = \left(X_0^{1/3} + \tfrac{1}{3}\,W(t)\right)^3.$$

(10)

3. Apply the Euler–Maruyama method

$$X_{j+1} = X_j + f(X_j)\,\Delta t + g(X_j)\,\Delta W_j$$

to the stochastic differential equation

$$\mathrm{d}X = \mu\, X\,\mathrm{d}W\,.$$

Show that

$$\mathbb{E}\left[|X_{j+1}|^2\right] = \left(1 + \Delta t\,|\mu|^2\right)\mathbb{E}\left[|X_j|^2\right].$$

(10)

4. On the interval $[0, 2]$, consider the boundary value problem

$$-y''(x) = f(x)\,,$$
$$y'(0) = y'(2) = 0\,.$$

(a) Away from the boundary, the solution is approximated by

$$-\frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} = f_j \,.$$

Show that the local truncation error of this method is of order 2.

(b) We approximate the boundary conditions by

$$\frac{y_1 - y_0}{h} = 0 \,,$$

with a corresponding expression for the second boundary condition. Show that this approximation is accurate only to order 1.

(c) Suggest an improvement that ensures the method is of order 2 up to the boundary.

(d) Write out the resulting system of linear equations of your method, or of the method given in (a) and (b), with only three nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. Is the resulting matrix invertible?

(e) Do you expect the solution to the original problem to be unique? Explain. (Think of what happens to constants. . .)

(f) **Extra credit:** Can you think of a reasonable condition that would make the solution unique? Note that you must not add a third boundary condition, as it would generically overdetermine the system.

(5+5+5+5+5+*10*)

5. Recall that the Householder reflector about the hyperplane normal to $\boldsymbol{v}$ is the matrix

$$H = I - 2\,\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} \,.$$

(a) What are the eigenvalues of H?
Hint: What do you get by applying H to $\boldsymbol{v}$, or to a vector orthogonal to $\boldsymbol{v}$?

(b) Find $\boldsymbol{v}$ and α such that

$$H \begin{pmatrix} 0 \\ 4 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix} \,.$$

(10+10)

6. You minimize the function

$$f(x, y) = x^2 - y^2$$

where x and y are constrained to the unit circle, i.e.

$$h(x, y) = x^2 + y^2 - 1 = 0 \,.$$

(a) State the exact solution to this problem.
(No computation required, the problem is simple enough to spot the answer.)

(b) Solve the problem using the quadratic penalty method, i.e. minimize

$$p_\alpha(x, y) = f(x, y) + \alpha\, h^2(x, y)\,.$$

Compute the minimizer (x_α^*, y_α^*) of the penalized problem explicitly.

(c) Show that $h(x_\alpha^*, y_\alpha^*) \to 0$ as $\alpha \to \infty$.

(10+10+5)