

Engineering and Science Mathematics 2B

Homework 3

due February 25, 2004, before 12:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

- Determine if the following are vector spaces. If not, explain which property fails.
 - The polynomials of degree smaller or equal to n , with the usual addition and multiplication by a scalar.
 - The polynomials of degree n , with the usual addition and multiplication by a scalar.
 - The set of $n \times m$ matrices with the usual addition and multiplication by a scalar.
 - The set of $n \times m$ matrices with matrix multiplication taking the role of vector addition, scalar multiplication being as usual.
 - The set of symmetric $n \times n$ matrices with the usual addition and multiplication by a scalar.
 - The set of invertible $n \times n$ matrices with the usual addition and multiplication by a scalar.
 - Complex numbers form a vector space over the reals of dimension 2.
 - Complex numbers form a vector space over the complex numbers of dimension 2.
- Let $\mathbf{v} = (1, 2, 3)^T$ be a vector expressed in coordinates with respect to the standard basis of \mathbb{R}^3 . Find the coordinates of this vector with respect to the basis

$$\mathbf{b}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

- (E) Determine whether the following vectors form a basis of \mathbb{R}^4 . If not, obtain a basis by adding and/or removing vectors from the set.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

- (A) Let V be a vector space, $\mathbf{b}_1, \dots, \mathbf{b}_n$ a basis of V , and $\mathbf{v}_1, \dots, \mathbf{v}_m$ where $m \leq n$ a set of linearly independent vectors in V . Show that you can construct another basis for V consisting of $\mathbf{v}_1, \dots, \mathbf{v}_m$ and $n - m$ vectors from among $\mathbf{b}_1, \dots, \mathbf{b}_n$.
Hint: Successively replace one of the \mathbf{b}_i by a vector \mathbf{v}_j .

4. Use the definition of the matrix inverse to show that $(AB)^{-1} = B^{-1}A^{-1}$.
5. Use the method taught in class to compute the inverse of

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

6. (E) Is the following matrix invertible? If yes, compute its inverse.

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- (A) Prove that the following are equivalent.
- (a) $A \in M(n \times n)$ is invertible.
 - (b) $\text{Ker } A = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = 0\}$ contains only the zero vector.
 - (c) The columns of A are linearly independent.