Engineering and Science Mathematics 2B

Review for Midterm I

March 3, 2004, 08:15–9:30

1. Equations for lines and planes; distance of a point to a line or plane; distance between two lines. See, in particular, the examples on pp. 234–237.

2. Complex Numbers: Know how to do arithmetic with complex numbers; polar representation of complex numbers; complex logarithm.

3. Solve a system of linear equations: See handout. Practice problem:

$$A = \begin{pmatrix} 1 & 2 & 5 & 1 & 0 \\ -1 & -1 & -4 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 1 & 2 & 5 & 0 & -1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

   (a) Solve $A\boldsymbol{x} = \boldsymbol{b}$.

   (b) Characterize all vectors $\boldsymbol{b}$ for which the equation has a solution.

   (c) Find a basis for the kernel of A.

   (d) Find a basis for the range of A.

4. Concept of vector space, linear independence, basis.

5. Matrix inversion: see handout and examples from homework.

6. Linear transformations: Definition, representation by a matrix, change of basis. Practice problem:

   Let
$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix},$$
   and
$$\boldsymbol{w}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad \boldsymbol{w}_2 = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}, \quad \boldsymbol{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$
   be two bases in $\mathbb{R}^3$.

(a) Write the vector $\boldsymbol{u} = (2,3,5)^T$ in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(b) Find the change of basis matrix from the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ to the basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$.

(c) Let $\boldsymbol{a} = 4\boldsymbol{v}_1 - 2\boldsymbol{v}_2 + \boldsymbol{v}_3$; find its coordinates in the basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$.

(d) Let the linear transformation F be the reflection about the plane spanned by the standard unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$; find the matrix representing F in the standard basis and in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(e) Compute $F(\boldsymbol{a})$ in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(f) Let $\boldsymbol{b} = \boldsymbol{w}_1 + 2\boldsymbol{w}_2 + 3\boldsymbol{w}_3$; compute $F(\boldsymbol{b})$ in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

7. Determinants. See, for example, p. 312 question 8.2. Part (b) was done in class, the solution given in the book is wrong!

8. Eigenvalues and eigenvectors. Practice problem: Diagonalize the matrix

$$B = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}.$$

9. Test your understanding: Are the following statements true or false? If false, give a short argument.

(a) If A is an $n \times k$ matrix with $n > k$, then its columns are linearly independent.

(b) If A is an $n \times k$ matrix with $n < k$, then its columns are linearly dependent.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions if the nullspace of A is nontrivial.

(d) Suppose that A is invertible. Then A^T is also invertible and its inverse is the transpose of A^{-1}.

(e) There exists such a 3×3 matrix A that $\operatorname{Range} A = \operatorname{Ker} A$

(f) The set of orthogonal 3×3 matrices forms a vectorspace with the usual matrix addition and scalar multiplication. (Recall that a matrix is orthogonal if $A^T = A^{-1}$.)

(g) The projection onto the plane $x + 3y - 2z = 1$ in $\mathbb{R}^3$ is a linear transformation.

(h) Two eigenvectors of a matrix A are always linearly independent.

(i) Every square matrix is diagonalizable.

(j) Every regular square matrix is diagonalizable

(k) Every hermitian matrix is diagonalizable.