Engineering and Science Mathematics 2B

Midterm II

March 31, 2004

For questions with an easy (E) and an advanced (A) version, choose either the easy or the advanced version, not both. You may make a different choice for each question.
Useful identities:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$
$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$
$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$
$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x}\, f(x)\, dx$$
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\, \tilde{f}(\xi)\, d\xi$$
$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\, d\xi$$

1. (E) Define an inner product on $\mathbb{R}^2$ by
$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^T A \boldsymbol{v} \qquad \text{with} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$
Are the vectors
$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
orthogonal with respect to this inner product? Explain. (8)

   (A) Show that $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^H A \boldsymbol{v}$ defines an inner product on $\mathbb{C}^n$ provided that A is a Hermitian $n \times n$ matrix, and provided that all eigenvalues of A are strictly positive. (10)

2. (E) Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by
$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$
(8)

(A) Find an orthonormal basis for the subspace of $C[0,\pi]$ spanned by

$$f_1(x) = \sin^2 x\,, \qquad f_2(x) = \cos^2 x\,,$$

with respect to the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\, g(x)\, \mathrm{d}x\,.$$

You do not need to explicitly compute the normalization of the second basis function. (10)

3. (E) Show that the Fourier coefficient with wavenumber $k = 1$ of the function

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi \\ x - \pi & \text{for } \pi \le x < 2\pi \end{cases}$$

is zero. (8)

(A) Let f be a π-periodic function on the interval $[0, 2\pi]$, i.e. $f(x) = f(x+\pi)$. Show that the Fourier coefficient $f_k = 0$ if k is odd.

(10)

4. (a) Derive the Parseval identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(\xi)|^2\, \mathrm{d}\xi\,.$$

(b) Show that the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise}\,. \end{cases}$$

is given by

$$\tilde{f}(\xi) = \sqrt{\frac{2}{\pi}}\, \frac{\sin \xi}{\xi}\,.$$

(c) Use (a) and (b) to prove

$$\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2}\, \mathrm{d}\xi = \pi\,.$$

(10+10+10)

5. Let f be a real-valued odd function, i.e. $f(-x) = -f(x)$.

(a) Show that the Fourier transform $\tilde{f}$ is also odd.

(b) Show that the Fourier transform $\tilde{f}$ is purely imaginary.

(10+10)