Engineering and Science Mathematics 2B

Solutions to Midterm II

Sönke Mittwollen

1. (E) Define an inner product on $\mathbb{R}^2$ by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T A \boldsymbol{v} \qquad \text{with} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Are the vectors

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

orthogonal with respect to this inner product? Explain.

(A) Show that $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^H A \boldsymbol{v}$ defines an inner product on $\mathbb{C}^n$ provided that A is a Hermitian $n \times n$ matrix, and provided that all eigenvalues of A are strictly positive.

Answer:

(E) Two vectors are orthogonal with respect to an inner product iff the inner product of the two vectors is zero.

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^T A \boldsymbol{w} = (1,1) \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1,1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \neq 0\,,$$

therefore the two vectors are not orthogonal.

(A) What has to be shown:

(a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle^*$

(b) $\langle \boldsymbol{u}, \lambda \boldsymbol{v} + \mu \boldsymbol{w} \rangle = \lambda \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \mu \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

(c) $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ iff $\boldsymbol{u} = 0$, and $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$ otherwise.

Solution:

(a) For complex scalars, the Hermitian conjugate equals the complex conjugate, so that

$$\begin{aligned}
\langle \boldsymbol{v}, \boldsymbol{u} \rangle^* &= \langle \boldsymbol{v}, \boldsymbol{u} \rangle^H \\
&= (\boldsymbol{v}^H A \boldsymbol{u})^H \\
&= \boldsymbol{u}^H A^H (\boldsymbol{v}^H)^H \\
&= \boldsymbol{u}^H A^H \boldsymbol{v} \\
&= \boldsymbol{u}^H A \boldsymbol{v}, \text{ since } A \text{ is Hermitian} \\
&= \langle \boldsymbol{u}, \boldsymbol{v} \rangle
\end{aligned}$$

(b) By direct computation,

$$\begin{aligned}\langle \boldsymbol{u}, \lambda\boldsymbol{v}+\mu\boldsymbol{w}\rangle &= \boldsymbol{u}^H A(\lambda\boldsymbol{v}+\mu\boldsymbol{w}) = \boldsymbol{u}^H(\lambda A\boldsymbol{v}+\mu A\boldsymbol{w}) = \lambda\boldsymbol{u}^H A\boldsymbol{v}+\mu\boldsymbol{u}^H A\boldsymbol{w}\\ &= \lambda\langle \boldsymbol{u},\boldsymbol{v}\rangle + \mu\langle \boldsymbol{u},\boldsymbol{w}\rangle\end{aligned}$$

(c) Since $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\}$ form a basis, an arbitrary vector $\boldsymbol{w}$ can be written

$$\boldsymbol{w} = \sum_{k=1}^{n} \alpha_k\,\boldsymbol{v}_k\,,$$

and we see that

$$\langle \boldsymbol{w},\boldsymbol{w}\rangle = \sum_{k=1}^{n} \lambda_k\,|\alpha_k|^2\,.$$

Since $\lambda_k > 0$, this expression is always non-negative, and is zero only if all α_k are zero, i.e. if $\boldsymbol{w} = 0$.

2. (E) Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix}.$$

(A) Find an orthonormal basis for the subspace of $C[0,\pi]$ spanned by

$$f_1(x) = \sin^2 x\,, \qquad f_2(x) = \cos^2 x\,,$$

with respect to the inner product

$$\langle f,g\rangle = \int_0^{2\pi} f(x)\,g(x)\,\mathrm{d}x\,.$$

Answer:

(E) Using the Gram–Schmidt procedure, we get:

$$\begin{aligned}\boldsymbol{n}_1 &= \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\\ \boldsymbol{b}_2 &= \boldsymbol{v}_2 - \langle \boldsymbol{b}_1,\boldsymbol{v}_2\rangle\boldsymbol{b}_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{2}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}\\ \boldsymbol{n}_2 &= \frac{1}{\sqrt{1+4+1}}\begin{pmatrix}1\\-2\\1\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}\end{aligned}$$

Now $\{\boldsymbol{n}_1,\boldsymbol{n}_2\}$ is an orthonormal basis of the given subspace.

(A)

$$\|f_1(x)\| = \int_0^{2\pi} f^2(x)\,dx = \int_0^{2\pi} \sin^4 x\,dx$$

With the given formulas we find

$$\sin^4 x = \Big(\frac{1-\cos 2x}{2}\Big)^2 = \frac{1 - 2\cos 2x + \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}(1+\cos 4x)$$
$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\,,$$

so we get

$$\|f_1(x)\| = \int_0^{2\pi} \frac{3}{8}\,dx - \int_0^{2\pi} \frac{1}{2}\cos 2x\,dx + \int_0^{2\pi} \frac{1}{8}\cos 4x\,dx = 2\pi\frac{3}{8} = \frac{3}{4}\pi$$

$$n_1(x) = \frac{f_1(x)}{\|f_1(x)\|} = \frac{2}{\sqrt{3\pi}}\sin^2 x$$

$$b_2(x) = f_2(x) - \langle n_1(x), f_2(x)\rangle n_1(x) = f_2(x) - \frac{2}{3\pi}\langle f_1(x), f_2(x)\rangle f_1(x)$$

$$\langle f_1(x), f_2(x)\rangle = \int_0^{2\pi} \sin^2 x \cos^2 x\,dx = \int_0^{2\pi} \frac{1-\cos 2x}{2}\frac{1-\cos 2x}{2}\,dx$$
$$= \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{4}\cos^2 2x\right)dx = \int_0^{2\pi} \left(\frac{1}{4} - \frac{1+\cos 4x}{8}\right)dx$$
$$= \frac{\pi}{4}$$

$$b_2(x) = \cos^2 x - \frac{\pi}{4}\,\frac{4}{3\pi}\,\sin^2 x = \cos^2 x - \frac{1}{3}\,(1-\cos^2 x) = \frac{4}{3}\cos^2 x - \frac{1}{3}$$

The following normalization was not needed

$$\|b_2(x)\| = \int_0^{2\pi} \left(\frac{4}{3}\cos^2 x - \frac{1}{3}\right)^2 dx = \int_0^{2\pi} \left(\frac{16}{9}\cos^4 x - \frac{8}{9}\cos^2 x + \frac{1}{9}\right) dx$$
$$= \int_0^{2\pi} \left(\frac{16}{9}\Big(\frac{1+\cos 2x}{2}\Big)^2 - \frac{8}{9}\frac{1+\cos 2x}{2} + \frac{1}{9}\right) dx$$
$$= \int_0^{2\pi} \left(\frac{4}{9}(1 + 2\,\cos 2x + \cos^2 2x) + \frac{4}{9}\,(1+\cos 2x) + \frac{1}{9}\right) dx$$
$$= \int_0^{2\pi} \left(\frac{1}{9} + \frac{4}{9}\cos 2x + \frac{4}{9}\frac{1+\cos 4x}{2}\right) dx$$
$$= \frac{2\pi}{3}$$

$$n_2(x) = \frac{1}{\sqrt{6\pi}}\,(4\,\cos^2 x - 1)$$

3. (E) Show that the Fourier coefficient with wavenumber $k = 1$ of the function

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi \\ x - \pi & \text{for } \pi \le x < 2\pi \end{cases}$$

is zero.

(A) Let f be a π-periodic function on the interval $[0, 2\pi]$, i.e. $f(x) = f(x + \pi)$. Show that the Fourier coefficient $f_k = 0$ if k is odd.

Answer:

(E)

$$\sqrt{2\pi}\, f_k = \int_0^{2\pi} e^{ikx}\, f(x)\, dx = \int_0^{\pi} e^{ikx}\, x\, dx + \int_{\pi}^{2\pi} e^{ikx}\, (x - \pi)\, dx$$

Substitution $y = x - \pi$ and $dx = dy$ gives

$$= \int_0^{\pi} e^{ikx}\, x\, dx + \int_0^{\pi} e^{ik(y+\pi)}\, y\, dy$$

$$= \int_0^{\pi} e^{ikx}\, x\, dx + e^{ik\pi} \int_0^{\pi} e^{iky}\, y\, dy$$

$= 0$ for $k = 1$, because $e^{i\pi} = -1$

(A)

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$$= \int_0^{\pi} e^{ikx}\, f(x)\, dx + e^{ik\pi} \int_0^{\pi} e^{iky}\, f(y)\, dy$$

$= 0$ for odd k, because $e^{i\pi(2m+1)} = -1$

4. (a) Derive the Parseval identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(\xi)|^2\, \mathrm{d}\xi\, .$$

(b) Show that the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise}\, . \end{cases}$$

is given by

$$\tilde{f}(\xi) = \sqrt{\frac{2}{\pi}}\, \frac{\sin \xi}{\xi}\, .$$

(c) Use (a) and (b) to prove

$$\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2}\, \mathrm{d}\xi = \pi\,.$$

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2\, dx &= \int_{-\infty}^{\infty} f^*(x) f(x)\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta x}\, \tilde{f}^*(\eta)\, d\eta \int_{-\infty}^{\infty} e^{i\xi x}\, \tilde{f}(\xi)\, d\xi\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi-\eta)x}\, dx\ \tilde{f}^*(\eta) \tilde{f}(\xi)\, d\eta\, d\xi \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\xi - \eta)\ \tilde{f}^*(\eta) \tilde{f}(\xi)\, d\eta\, d\xi = \int_{-\infty}^{\infty} \tilde{f}^*(\xi) \tilde{f}(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} |\tilde{f}(\xi)|^2\, d\xi
\end{aligned}$$

(b)

$$\begin{aligned}
\tilde{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x}\, f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \frac{-1}{i\xi} (e^{-i\xi} - e^{i\xi}) \\
&= \frac{1}{\sqrt{2\pi}} \frac{2}{\xi} \frac{e^{i\xi} - e^{-i\xi}}{2i} = \sqrt{\frac{2}{\pi}} \frac{\sin \xi}{\xi}
\end{aligned}$$

(c)

$$\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2}\, \mathrm{d}\xi = \frac{\pi}{2} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin \xi}{\xi} \right)^2 \mathrm{d}\xi \overset{(a),(b)}{=} \frac{\pi}{2} \int_{-1}^{1} 1^2\, \mathrm{d}x = \pi$$

5. Let f be a real-valued odd function, i.e. $f(-x) = -f(x)$.

   (a) Show that the Fourier transform $\tilde{f}$ is also odd.

   (b) Show that the Fourier transform $\tilde{f}$ is purely imaginary.

Answer:

(a)

$$\tilde{f}(-\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-\xi)x}\, f(x)\, dx \tag{1}$$

$$\text{Substitution } y = -x \text{ and } \mathrm{d}x = -\mathrm{d}y \tag{2}$$

$$\begin{aligned}
&= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{i\xi(-y)}\, f(-y)\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi y}\, (-f(y))\, dy \\
&= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi y}\, f(y)\, dy = -\tilde{f}(\xi)
\end{aligned}$$

(b) We have to show tht $\tilde{f}^*(\xi) = -\tilde{f}(\xi)$.

$$\begin{aligned}
\tilde{f}^*(\xi) &= \Big(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx\Big)^* \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} f^*(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x}\, f(x)\, dx \quad \text{since } f \text{ is real; this is the RHS of (1)} \\
&= -\tilde{f}(\xi)
\end{aligned}$$