1. (E) Find the distance of the point $p = (0,1,0)^T$ to the plane

$$(x - a) \cdot n = 0$$

where $a = (1,0,1)^T$ and $n = \frac{1}{\sqrt{3}}(1,-1,-1)^T$.

(8)

(A) Show that two lines in $\mathbb{R}^3$ lie in a plane only if either they intersect or they are parallel. (10)

(E) $d = |(p - a) \cdot \hat{n}|$

$$= \left| \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right| = \frac{1}{\sqrt{3}}$$

(A) Assume the two lines are described by the parametric equations

$$x = a + \lambda v$$
$$x = b + \mu w$$

If the two lines lie in a plane, then the vectors

$$v, w, a - b$$

are linearly dependend. I.e. there is α, β, γ not all zero such that

$$\alpha v + \beta w + \gamma(a - b) = 0 \qquad (*)$$

If $\gamma = 0$, then $\alpha = -\beta$, i.e$_2$ the lines are parallel.

If $\gamma \neq 0$, set $\lambda = \frac{\alpha}{\gamma}$ and $\mu = -\frac{\beta}{\gamma}$. Then (*) shows that the two lines intersect in the point $a + \lambda v = b + \mu w$.

□

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2. Find the general solution to the system of linear equations $Ax = b$ with

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & -2 & -3 & -4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ -10 \\ 4 \end{pmatrix}$$

Check your answer! (10)

3. (E) Consider the matrix A from the previous question. Show that $Ax = c$ does not have a solution when

$$c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(8)

(A) Consider the matrix A from the previous question. Characterize all vectors $c \in \mathbb{R}^3$ such that $Ax = c$ is solvable. (10)

2. $$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ -1 & -2 & -3 & -4 & -10 \\ 1 & 1 & 1 & 1 & 4 \end{array}\right) \xrightarrow{\substack{R3 \to R1 \\ R1 \to R2 \\ R2+R3 \to R3}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & -1 & -2 & -3 & -6 \end{array}\right)$$

$$\xrightarrow{\substack{R1-R2 \to R1 \\ R2+R3 \to R3}} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow x = \begin{pmatrix} -2 \\ 6 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 3 \\ 0 \\ -1 \end{pmatrix}$$

Check: $$A \begin{pmatrix} -2 \\ 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + 6 + 0 + 0 \\ 2 - 12 + 0 + 0 \\ -2 + 6 + 0 + 0 \end{pmatrix} = b \quad ✓$$

$$A \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + 2^{3} - 2 + 0 \\ 1 - 4 + 3 + 0 \\ -1 + 2 - 1 + 0 \end{pmatrix} = 0 \quad ✓$$

$$A \begin{pmatrix} -2 \\ 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 + 3 + 0 - 3 \\ 2 - 6 + 0 + 4 \\ -2 + 3 + 0 - 1 \end{pmatrix} = 0 \quad ✓$$

3.

(E) $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 1 \\ -1 & -2 & -3 & -4 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -3 & 0 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

$\Rightarrow$ The system is inconsistent.

(A) Solution 1: Take a general vector c on the right:

$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & c_1 \\ -1 & -2 & -3 & -4 & c_2 \\ 1 & 1 & 1 & 1 & c_3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & c_3 \\ 0 & 1 & 2 & 3 & c_1 \\ 0 & -1 & -2 & -3 & c_2+c_3 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & c_3 \\ 0 & 1 & 2 & 3 & c_1 \\ 0 & 0 & 0 & 0 & c_1+c_2+c_3 \end{array}\right)$

$\Rightarrow$ For consistency, we need $c_1+c_2+c_3 = 0$

Solution 2: The system is solvable if c is in the range of A. So the task is to find a basis for the range, which is easily seen to be

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

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4. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Write out a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$.

(c) Check your result by explicitly performing the matrix multiplications SD and AS.

(10+5+5)

(a) $0 \equiv p_A(\lambda) = \det(A-\lambda I)$

$= \begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 - 1 \quad = \big((1-\lambda)-1\big)\big((1-\lambda)+1\big)$

$= -\lambda(2-\lambda)$

$\Rightarrow \lambda_1 = 0, \ \lambda_2 = 2$

Eigenvector for λ_1: Need $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} v_1 = 0$

$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \xrightarrow{R2+iR1 \to R2} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad v_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$

Eigenvector for λ_2: Need $\begin{pmatrix} 1-2 & i \\ -i & 1-2 \end{pmatrix} v_2 = 0$

$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \xrightarrow[R2-iR1 \to R2]{-R1 \to R1} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -i \\ -1 \end{pmatrix}$

(b) We see from (a) that $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $S = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}$

(c)

$$SD = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2i \\ 0 & -2 \end{pmatrix}$$

$$AS = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} i-i & -i-i \\ 1-1 & -1-1 \end{pmatrix} = SD$$ ✓

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5. (E) Give an example of two matrices A and B such that $AB \neq BA$. (An explicit calculation is required!) (8)

   (A) Let $A, B \in M(n \times n)$ such that there exists a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ that is also a set of eigenvectors for both A and B. Show that, in this case, $AB = BA$. (10)

(E) For example,

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq AB$$

(A) If $S = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, then

$$C = S^{-1} A S$$

and $D = S^{-1} B S$

are diagonal matrices, so that $CD = DC$

$$\Rightarrow AB = SCS^{-1} SDS^{-1} = SCDS^{-1} = SDCS^{-1}$$

$$= SDS^{-1} SCS^{-1}$$

$$= BA$$ □

6. (E) Let

$$v_1 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

(a) Let the linear transformation F be the reflection about the plane spanned by the standard unit vectors e_1 and e_2. Find the matrix representing F in the standard basis.

(b) Check that $\{v_1, v_2, v_3\}$ form a basis of $\mathbb{R}^3$.

(c) Find the matrix representing F in the basis $\{v_1, v_2, v_3\}$.

(d) Let $a = 4v_1 - 2v_2 + v_3$; compute $F(a)$ in the basis $\{v_1, v_2, v_3\}$.

(5+5+9+5)

(a) Clearly $M^F_{E,E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(e_1 and e_2 are left untouched, e_3 is normal to the plane and hence being reflected by having its direction reversed.)

(b,c)
$$\left(\underbrace{\begin{matrix} 2 & 2 & 3 \\ -2 & 1 & -1 \\ -1 & -2 & -2 \end{matrix}}_{\equiv S} \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right.\right) \xrightarrow[\substack{R1+2R3 \to R3}]{\substack{-R3 \to R1 \\ R1+R2 \to R2}} \left(\begin{matrix} 1 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{matrix} \left| \begin{matrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{matrix} \right.\right)$$

$$\xrightarrow[R2+\frac{3}{2}R3 \to R3]{\substack{R1+R3 \to R1 \\ -\frac{1}{2}R3 \to R2}} \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{matrix} \left| \begin{matrix} 1 & 0 & 1 \\ -\frac{1}{2} & 0 & -1 \\ \frac{5}{2} & 1 & 3 \end{matrix} \right.\right) \xrightarrow[2R3 \to R3]{\substack{R1-2R3 \to R1 \\ R2-R3 \to R2}} \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \underbrace{\begin{matrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{matrix}}_{= S^{-1}} \right.\right)$$

clearly $V = \{v_1, v_2, v_3\}$ is l.i., hence a basis of $\mathbb{R}^3$, and S represents the change of basis from V to E.

$$\Rightarrow M^F_{V,V} = S^{-1} M^F_{E,E} S$$

$$= \begin{pmatrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ -2 & 1 & -1 \\ -1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} -4 & -2 & -5 \\ -3 & -1 & -4 \\ 5 & 2 & 6 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ -2 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -9+4-5 & -8-2-10 & -12+2-10 \\ -6+2-4 & -6-1-8 & -9+1-8 \\ 10-4+6 & 10+2+12 & 15-2+12 \end{pmatrix} = \begin{pmatrix} -9 & -20 & -20 \\ -8 & -15 & -16 \\ 12 & 24 & 25 \end{pmatrix}$$

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d)

$$M^F_{V,V} \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -16 \\ -18 \\ 25 \end{pmatrix}$$

(A) Let P_2 be the vector space of polynomials of degree less or equal than 2.

(a) Show that

$$F(p) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$$

is a linear transformation from P_2 into $\mathbb{R}^3$.

(b) Find the matrix representing F when P_2 is endowed with basis $B = \{1, x, x^2\}$, and $\mathbb{R}^3$ is endowed with its standard basis.

(c) Find the matrix representing

$$G(p) = \begin{pmatrix} p'(0) \\ p'(1) \\ p'(2) \end{pmatrix}$$

with respect to the same pair of bases.

(d) Is F invertible? Is G invertible? Explain.

(e) Find a matrix A such that

$$A \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} = \begin{pmatrix} p'(0) \\ p'(1) \\ p'(2) \end{pmatrix}$$

for every $p \in P_2$.

(5+5+5+5+10)

(a) $$F(p+q) = \begin{pmatrix} p(0)+q(0) \\ p(1)+q(1) \\ p(2)+q(2) \end{pmatrix} = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} + \begin{pmatrix} q(0) \\ q(1) \\ q(2) \end{pmatrix} = F(p) + F(q)$$

$$F(\lambda p) = \begin{pmatrix} \lambda p(0) \\ \lambda p(1) \\ \lambda p(2) \end{pmatrix} = \lambda \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} = \lambda F(p)$$

b) $$M_F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$ (the colums are the images of the basis vectors)

c) $$M_G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

d) M_G is clearly singular, hence G is not invertible.

(Since $G = F \circ \frac{d}{dx}$, and $\operatorname{Ker} \frac{d}{dx} = \{1\}$, $\operatorname{Ker} G \supseteq \{1\}$ and therefore G cannot be invertible.)

For M_F:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

$$\underbrace{}_{= M_F^{-1}}$$

$\Rightarrow$ F is invertible

(e)

$$\begin{tikzcd} & P \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} \arrow[rr, "A"'] & & \begin{pmatrix} p'(0) \\ p'(1) \\ p'(2) \end{pmatrix} \end{tikzcd}$$

$$\Rightarrow A = M_G M_F^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{pmatrix}.$$