Engineering and Science Mathematics 2B

Final Exam

May 21, 2004

For questions with an easy (E) and an advanced (A) version, choose either the easy or the advanced version, not both. You may make a different choice for each question.

Useful identities:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\,, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cos^2\theta = \tfrac{1}{2}\left(1+\cos 2\theta\right), \qquad \sin^2\theta = \tfrac{1}{2}\left(1-\cos 2\theta\right)$$

$$f_k = \frac{1}{\sqrt{L}}\int_a^{a+L} \boldsymbol{e}^{-\frac{2\pi i k x}{L}}\, f(x)\,dx\,, \qquad f(x) = \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} f_k\,\boldsymbol{e}^{\frac{2\pi i k x}{L}}$$

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x}\, f(x)\,dx\,, \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\,\tilde{f}(\xi)\,d\xi$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\,d\xi$$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$P(A\cap B) = P(A)P(B|A)$$

$$P(A) = P(B)P(A|B) + P(\overline{B})P(A|\overline{B})$$

1. Compute the inverse of

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}.$$

(10)

2. Let V be the vector space of continuous functions on $\mathbb{R}$ spanned by the basis

$$E = \{\mathrm{e}^{ikx}, e^{-ikx}\}$$

with the usual addition and scalar multiplication.

(a) Let

$$T = \{\sin kx, \cos kx\}.$$

Explain why T is also a basis of V, and find the matrix S for the change of basis from T to E.

(Recall that $e^{ix} = \cos x + i \sin x$.)

(b) Let f be defined by the Fourier sine and cosine series

$$f(x) = \sum_{k=1}^{\infty} \left(2\,\frac{\cos kx}{k} + 2\,\frac{sinkx}{k} \right).$$

Use part (a) to compute the complex Fourier series for f.

(5+5)

3. (E) Show that the eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix are orthogonal. (8)

(A) Let A be a Hermitian matrix and $\boldsymbol{u}$ a vector with $||\boldsymbol{u}|| = 1$. Show that

$$\lambda_{\min} \leq \boldsymbol{u}^H A \boldsymbol{u} \leq \lambda_{\max}$$

Hint: Expand $\boldsymbol{u}$ in terms of a basis of orthonormal eigenvectors of A.

(10)

4. Compute the complex Fourier series of $f(x) = \cos\left(\frac{x}{2}\right)$ on the interval $[-\pi, \pi]$. (10)

5. Recall that the convolution of two functions ϕ and θ is defined

$$(\phi * \theta)(t) = \int_{-\infty}^{\infty} \phi(s)\,\theta(t-s)\,\mathrm{d}s.$$

Show that

$$\mathcal{F}(f \cdot g) = \frac{1}{\sqrt{2\pi}}\,\tilde{f} * \tilde{g}.$$

(10)

6. (E) A die is tossed. Let the random variable X denote the value shown. Define a second random variable by $Y = (X - \frac{7}{2})^2$. Give the probability functions of both X and Y.

(8)

(A) The die is tossed a large number of times N, each of the trials being denoted by a random variable X_i. The central limit theorem states that probability function of the random variable

$$X = \frac{X_1 + \cdots + X_N}{N}$$

asymptotes to a normal distribution with mean $\mu = \frac{7}{2}$ and variance $\sigma^2 = \sigma_i^2/N$. Compute the probability distribution function of $Y = (X - \frac{7}{2})^2$ in terms of μ and σ.

(10)

7. A chip manufacturer produces CPUs that are so complex that it is impossible to test every single function of the CPU in a production run, but it is possible to design a test that finds a defect with probability $p < 1$. Moreover, $\frac{9}{10}$ of all CPUs that are produced are defective. What value of p is needed to guarantee that 99% of all CPUs shipped work?

(10)

8. What is the probability of drawing exactly 3 red balls in 4 trials from a bag containing 4 red balls and 2 green ones, if

   (a) a ball drawn is replaced after each trial?

   (b) a ball once drawn is placed aside?

(5+5)

9. (E) Show that if X and Y are independent random variables, then

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]\,.$$

(8)

   (A) Use the moment generating function $M_X(t) = E[e^{tX}]$ to derive the mean and variance of the binomial distribution where

$$X = \text{number of successes in } n \text{ trials}\,,$$

   and the probability of success is p. (10)