Final Solutions

1.

$$\underbrace{\left(\begin{array}{ccc|ccc} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 1 & 0 & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & 1 \end{array}\right)}_{A} \to \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 3 & 0 & 0 \\ 2 & -2 & 1 & 0 & 3 & 0 \\ 2 & 1 & -2 & 0 & 0 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 3 & 0 & 0 \\ 0 & -6 & -3 & -6 & 3 & 0 \\ 0 & -3 & -6 & -6 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 3 & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 3 & 2 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{array}\right)$$

where the right-hand block of the last matrix is A^{-1}.

So this is an example where $A = A^{-1}$.

2. (a) T and E have the same number of l.i. vectors. Moreover

$$\sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\Rightarrow S = \begin{pmatrix} \frac{1}{2i} & \frac{1}{2} \\ -\frac{1}{2i} & \frac{1}{2} \end{pmatrix}$$ is the change-of-basis matrix from T to E.

(b) For $k \geq 1$: $a_k = \frac{2}{k}$, $b_k = \frac{2}{k}$

$$\Rightarrow \begin{pmatrix} c_k \\ c_{-k} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$$

$\Rightarrow$ $c_0 = 0$ and $c_k = \frac{1-i}{k}$ for $k \geq 1$ and $c_{-k} = c_k^*$.

3. (E) Suppose $Av = \lambda v$, $Aw = \mu w$ with $\mu \neq \lambda$

$$\Rightarrow \lambda^* v^H w = (\lambda v)^H w = (Av)^H w = v^H A^H w$$

$$= v^H A w = \mu v^H w$$

$\Rightarrow$ either $v^H w = 0$ or $\lambda^* = \mu$.

But Hermitian matrices have real eigenvalues, so $v^H w = 0$.

(A) Since A Hermitian, there exists an ONB of eigenvectors of A, $\{e_1, \dots, e_n\}$, say. Write

$$u = \sum_{i=1}^{n} u_i e_i$$

so that $\|u\|^2 = \sum_{i=1}^{n} |u_i| = 1$.

Similarly,

$$u^H A u = \sum_{i=1}^{n} \lambda_i |u_i|^2 \leq \lambda_{max} \sum_{i=1}^{n} |u_i|^2 = \lambda_{max}$$

and

$$u^H A u = \sum_{i=1}^{n} \lambda_i |u_i|^2 \geq \lambda_{min} \sum_{i=1}^{n} |u_i|^2 = \lambda_{min} .$$

4.

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} \cos\frac{x}{2} \, dx$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{-ikx + \frac{ix}{2}} dx + \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{-ikx - \frac{ix}{2}} dx$$

$$= \frac{1}{4\pi} \frac{1}{i(\frac{1}{2}-k)} e^{i(\frac{1}{2}-k)x} \Big|_{-\pi}^{\pi} + \frac{1}{4\pi} \frac{1}{i(-\frac{1}{2}-k)} e^{i(-\frac{1}{2}-k)x} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi} \frac{(-1)^k}{i(\frac{1}{2}-k)} \left(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}\right) + \frac{1}{4\pi} \frac{-(-1)^k}{i(\frac{1}{2}+k)} \left(e^{-i\frac{\pi}{2}} - e^{+i\frac{\pi}{2}}\right)$$

$$= \frac{(-1)^k}{2\pi} \frac{+\sin\frac{\pi}{2}}{\frac{1}{2}-k} + \frac{(-1)^k}{2\pi} \frac{+\sin\frac{\pi}{2}}{\frac{1}{2}+k}$$

$$= \frac{1}{2\pi} \frac{(-1)^k}{\frac{1}{4}-k^2}$$

5. $$\mathcal{F}(fg)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, g(x)\, dx$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-i\xi x} \int_{-\infty}^{\infty} e^{i\eta x} \tilde{f}(\eta)\, d\eta \int_{-\infty}^{\infty} e^{i\zeta x} \tilde{g}(\zeta)\, d\zeta\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\eta) \tilde{g}(\zeta) \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\zeta+\eta-\xi)x}\, dx}_{=\delta(\zeta+\eta-\xi)}\, d\zeta\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\eta)\, \tilde{g}(\xi-\eta)\, d\eta$$

$$= \frac{1}{\sqrt{2\pi}} (\tilde{f} * \tilde{g})(\xi)$$

6. (E) Let p_i denote the probability function for X.
As the 6 outcomes $x_1 = 1, \dots, x_6 = 6$ are equally likely,
$p_i = \frac{1}{6}$ for $i = 1, \dots, 6$.

Set $$y_1 = \left(x_1 - \tfrac{7}{2}\right)^2 = \left(x_6 - \tfrac{7}{2}\right)^2 = \tfrac{25}{4}$$
$$y_2 = \left(x_2 - \tfrac{7}{2}\right)^2 = \left(x_5 - \tfrac{7}{2}\right)^2 = \tfrac{9}{4}$$
$$y_3 = \left(x_3 - \tfrac{7}{2}\right)^2 = \left(x_4 - \tfrac{7}{2}\right)^2 = \tfrac{1}{4}$$

are also equally likely as values taken by Y, its probability function is given by $q_i = \frac{1}{3}$ for $i = 1, \dots, 3$.

Remark: $E[X] = \frac{7}{2}$, so $E[Y] = \mathrm{Var}[X]$.

We could compute this quantity in two ways:

(a) $E[Y] = \sum_{i=1}^{3} q_i y_i = \frac{1}{3}\left(\frac{25}{4} + \frac{9}{4} + \frac{1}{4}\right) = \frac{1}{3}\frac{35}{4} = \frac{35}{12}$

(b) $\mathrm{Var}[X] = E[X^2] - E[X]^2$

$$= \frac{1}{6}(1+4+9+16+25+36) - \frac{49}{4}$$

$$= \frac{91}{6} - \frac{49}{4} = \frac{182}{12} - \frac{147}{12} = \frac{35}{12}$$

(A) Note that $E[X] = \frac{7}{2}$, so $X - \frac{7}{2}$ has PDF

$$f_{X-\frac{7}{2}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

Therefore,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X-\frac{7}{2}}(x)\, \delta(x^2 - y)\, dx$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, \delta(x^2 - y)\, dx$$

Now change variables:

$$u = x^2 - y \quad \Rightarrow \quad x = \sqrt{u+y}$$

and $du = 2x\,dx = 2\sqrt{u+y}\,dx$

$$\Rightarrow f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-y}^{\infty} e^{-\frac{1}{2}\frac{u+y}{\sigma^2}} \delta(u) \frac{du}{\sqrt{u+y}}$$

$$= \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{y}{2\sigma^2}}$$

Note: The quoted version of the CLT is from the book. It is sloppy about the fact that as $N \to \infty$, $\sigma^2 \to 0$ so that, technically speaking, $f_X \to \delta$ in the sense of distributions. To get a non-trivial, precisely stated limit, one has to "zoom into" the convergence by looking at $Z = \sqrt{N}(X - \mu)$ instead. Yet, the version in the book is useful in practice because it gives a good approximation to the PDF of a long finite sum of independent random variables. (This remark is not relevant for the computation at hand.)

7. Let T be the event that a CPU is tested defective,

D be the event that the CPU is truly defective.

Then $p = P(T \mid D)$,

$$P(D) = \frac{9}{10}, \qquad P(\bar{D}) = 1 - \frac{9}{10} = \frac{1}{10},$$

and we assume that the test does not yield false positives, i.e.

$$P(T \mid \bar{D}) = 0, \quad \text{so } P(\bar{T} \mid \bar{D}) = 1.$$

Then, by Bayes' theorem,

$$P(\bar{D} \mid \bar{T}) = \frac{P(\bar{D})\, P(\bar{T} \mid \bar{D})}{P(\bar{T})}$$

$$= \frac{P(\bar{D})\, P(\bar{T} \mid \bar{D})}{P(\bar{D})\, P(\bar{T} \mid \bar{D}) + P(D)\, P(\bar{T} \mid D)}$$

$$= \frac{\frac{1}{10} \cdot 1}{\frac{1}{10} \cdot 1 + (1-p)\frac{9}{10}} = \frac{1}{10 - 9p}$$

So to achieve $P(\bar{D} \mid \bar{T}) = \frac{99}{100}$, we must have

$$\frac{99}{100}(10 - 9p) = 1 \Rightarrow 10 - 9p = \frac{100}{99} \Rightarrow p = \frac{1}{9}\left(10 - \frac{100}{99}\right)$$

$$\Rightarrow p = \frac{890}{891}$$

8. (a) This is a binomial distribution with $p = \frac{4}{6} = \frac{2}{3}$,

$n = 4$, $k = 3$, so

$$P(X=3) = {}^4C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^1 = \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 1} \quad \frac{2\cdot 2\cdot 2}{3^4}$$

$$= \frac{32}{81}$$

(b) $$P(\text{3 red balls drawn}) = \frac{{}^4C_3 \ {}^2C_1}{{}^6C_4} = \frac{\frac{4!}{3!\,1!} \cdot \frac{2!}{1!\cdot 1!}}{\frac{6!}{4!\ 2!}}$$

$$= \frac{4\cdot 2\cdot 2}{5\cdot 6} = \frac{8}{15}$$

9. (E) $$\mathrm{Var}[X+Y] = E[(X+Y)^2] - E[X+Y]^2$$

$$= E[X^2] + 2E[XY] + E[Y^2]$$
$$- E[X]^2 - 2E[X]E[Y] - E[Y]^2$$

$$= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2$$

↑ since $E[XY] = E[X]\,E[Y]$ when X, Y indep.

$$= \mathrm{Var}[X] + \mathrm{Var}[Y]$$

(A) Let $X_i = \begin{cases} 1 & \text{if ith trial is a success} \\ 0 & \text{if ith trial is a failure} \end{cases}$

$\Rightarrow X = X_1 + \dots + X_n$

$\Rightarrow M_X(t) = E[e^{tX}]$

$= E[e^{t(X_1 + \dots + X_n)}]$

$= E[e^{tX_1} \cdot \dots \cdot e^{tX_n}]$

$= E[e^{tX_1}] \cdots E[e^{tX_n}]$ as X_i independent

$= E[e^{tX_1}]^n$ as X_i identically distr.

$= (pe^t + q)^n$

$\Rightarrow M_X'(t) = npe^t (pe^t + q)^{n-1}$

$M_X''(t) = npe^t (pe^t + q)^{n-1} + n(n-1)(pe^t)^2 (pe^t + q)^{n-2}$

$E[X] = M_X'(0) = npe^0 (pe^0 + q)^{n-1} = np$

$Var[X] = M_X''(0) - E[X]^2 = np + n(n-1)p^2 - (np)^2$

$= np - np^2 = npq$