Engineering and Science Mathematics 2B

Homework 4

due March 1, 2003, before 22:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Consider the vector space of functions that is spanned by the basis
$$B = \{\sin x, \cos x, \sin 2x, \cos 2x\}\,.$$
Find the matrix representing the derivative operator with respect to the basis B.

2. Recall the definitions of range and kernel of a linear map A on the vector space $\mathbb{R}^n$:
$$\operatorname{Range} A = \{A\boldsymbol{x} \colon \boldsymbol{x} \in \mathbb{R}^n\}$$
$$\operatorname{Ker} A = \{\boldsymbol{x} \in \mathbb{R}^n \colon A\boldsymbol{x} = 0\}$$
(E) Let
$$A = \begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 6 \\ 0 & 1 & 1 \end{pmatrix}$$
Find a basis for Range A and for Ker A.

(A) Prove that $\dim \operatorname{Ker} A + \dim \operatorname{Range} A = n$.

3. Let
$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$
represent a linear transformation on $\mathbb{R}^3$ with respect to the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Find the matrix A' which represents this transformation with respect to the new basis
$$\boldsymbol{e}_1' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_2' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_3' = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

4. Compute the determinant
$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

5. (E) Use the determinant to test if the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

represents an invertible linear transformation.

(A) Use the definition of the determinant to show that a matrix A is invertible if and only if and only if $\det A \neq 0$.

6. Show that a matrix A is invertible if and only if all the eigenvalues of A are nonzero.

(Recall that λ is an eigenvalue of A and $\boldsymbol{v}$ is the corresponding eigenvector if $A\boldsymbol{v} = \lambda\boldsymbol{v}$.)