Engineering and Science Mathematics 2B

Homework 3

due February 22, 2003, before 24:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Determine if the following are vector spaces. If not, explain which property fails.
   (a) The polynomials of degree smaller or equal to n, with the usual addition and multiplication by a scalar.
   (b) The polynomials of degree n, with the usual addition and multiplication by a scalar.
   (c) The set of $n \times m$ matrices with the usual addition and multiplication by a scalar.
   (d) The set of $n \times m$ matrices with matrix multiplication taking the role of vector addition, scalar multiplication being as usual.
   (e) The set of symmetric $n \times n$ matrices with the usual addition and multiplication by a scalar.
   (f) The set of invertible $n \times n$ matrices with the usual addition and multiplication by a scalar.

2. Let $\boldsymbol{v} = (1, 2, 3)^T$ be a vector expressed in coordinates with respect to the standard basis of $\mathbb{R}^3$. Find the coordinates of this vector with respect to the basis

$$\boldsymbol{b}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

3. (E) Determine whether the following vectors form a basis of $\mathbb{R}^4$. If not, obtain a basis by adding and/or removing vectors from the set.

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

(A) Let V be a vector space, $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ a basis of V, and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ where $m \le n$ a set of linearly independent vectors in V. Show that you can construct another basis for V consisting of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ and $n - m$ vectors from among $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$.

Hint: Successively replace one of the $\boldsymbol{b}_i$ by a vector $\boldsymbol{v}_j$.

4. Use the definition of the matrix inverse to show that $(AB)^{-1} = B^{-1}A^{-1}$.

5. Use the method taught in class to compute the inverse of

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

6. (E) Is the following matrix invertible? If yes, compute its inverse.

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

(A) Prove that the following are equivalent.

(a) $A \in M(n \times n)$ is invertible.

(b) $\operatorname{Ker} A = \{\boldsymbol{v} \in \mathbb{R}^n \colon A\boldsymbol{v} = 0\}$ contains only the zero vector.

(c) The columns of A are linearly independent.