1. Diagonalize the matrix

$$A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

I.e., find a diagonal matrix D and a change of coordinate S such that $D = S^{-1}AS$. (10)

Eigenvalues:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -i \\ i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda - 2)$$

$\Rightarrow \lambda_1 = 0 \qquad \lambda_2 = 2$

Eigenvectors:

① $(A - \lambda_1 I)v_1 = 0$

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \xrightarrow{-iR1+R2\to R2} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow v_1 = \begin{pmatrix} -i \\ -1 \end{pmatrix}$

② $(A - \lambda_2 I)v_2 = 0$

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \xrightarrow{iR1+R2\to R2} \begin{pmatrix} -1 & -i \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow v_2 = \begin{pmatrix} i \\ -1 \end{pmatrix}$

$\Rightarrow D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$S = \begin{pmatrix} -i & i \\ -1 & -1 \end{pmatrix}$

Check: $SD = \begin{pmatrix} 0 & 2i \\ 0 & -2 \end{pmatrix}$

$AS = \begin{pmatrix} -i+i & i+i \\ 1-1 & -1-1 \end{pmatrix}$

$\Rightarrow D = S^{-1}AS$

2. (E) Are the eigenvectors of the matrix in Question 1 orthogonal? Explain why you can answer this question without even computing the eigenvalues. (4)

(A) Let A be a Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$. Show that

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^H. \qquad (*)$$

(5)

(E) Yes. The matrix is Hermitian, i.e. $A = A^H$, and eigenvectors with different eigenvalues of a Hermitian matrix are always orthogonal.

(A) If A is a Hermitian matrix, we can choose $v_1, \ldots, v_n$ to be an orthonormal basis of $\mathbb{C}^n$.

A linear transformation (i.e. a matrix) is completely determined by how it acts on each basis vector. Hence, consider the j-th basis vector:

$A v_j = \lambda_j v_j \qquad$ (since v_j is also an eigenvect

on the other hand,

$$\left(\sum_{i=1}^{n} \lambda_i v_i v_i^H\right) v_j = \sum_{i=1}^{n} \lambda_i v_i \underbrace{v_i^H v_j}_{=\delta_{ij}} = \lambda_j v_j$$

Hence, left and right side of (*) coincide when applied to any v_j $\Rightarrow$ left and right sides of (*) are equal. □

3. (E) Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(4)

(A) Find a basis of $\mathbb{R}^2$ which is orthonormal with respect to the non-standard inner product

$$\langle u, v \rangle = u^T A v \qquad \text{with } A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

(5)

(E) $$b_1 := \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 := v_2 - \langle b_1, v_2 \rangle \, b_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3}(1-1+1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{2}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$b_2 := \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\{b_1, b_2\}$ is a possible ONB (other answers are possible)

(A) Start with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and normalize:

$$\|e_1\|^2 = e_1^T A e_1 = (1 \;\; 0) \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (1 \;\; 0) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2$$

$$\Rightarrow \quad b_1 := \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 := e_2 - \langle b_1, e_2 \rangle e$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \underbrace{\frac{1}{2} (1 \;\; 0) \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{= \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}_{=1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\|u_2\|^2 = u_2^T A u_2 = \left(-\tfrac{1}{2} \;\; 1\right) \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$= \left(-\tfrac{1}{2} \;\; 1\right) \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}$$

$$\Rightarrow \quad b_2 := \frac{u_2}{\|u_2\|} = \sqrt{2} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix}$$

$\{b_1, b_2\}$ is a basis orthonormal w.r.t. the non-standard inner product.

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$= (1 \;\; 0) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2$

$\Rightarrow b_1 := \dfrac{e_1}{\|e_1\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u_2 := e_2 - \langle b_1, e_2 \rangle e$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \underbrace{\frac{1}{2} (1 \;\; 0) \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{= \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}_{= 1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

$\|u_2\|^2 = u_2^T A u_2 = \left(-\frac{1}{2} \;\; 1\right) \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

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$\{b_1, b_2\}$ is a basis othonormal w.r.t. the non-standard inner product.

4. (E) Let A be a Hermitian matrix, i.e. $A = A^H$, and consider the standard inner product where $\langle u, v\rangle = u^H v$. Show that

$$\langle u, Av\rangle = \langle Au, v\rangle .$$

(4)

(A) Consider the vector space of bounded differentiable functions with bounded first derivatives which, moverover, satisfy $f(0) = 0$. On this vector space we define the inner product

$$\langle f, g\rangle = \int_0^\infty f^*(x)\, g(x)\, e^{-x}\, dx .$$

Show that the operator

$$\mathcal{L}f = e^x \frac{d}{dx}\left(e^{-x}\frac{df}{dx}\right)$$

is Hermitian, i.e. that $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$. (5)

(E) $\langle u, Av\rangle = u^H Av = u^H A^H v = (Au)^H v = \langle Au, v\rangle$ □

(A) $$\langle f, \mathcal{L}g\rangle = \int_0^\infty f^*(x)\, e^x \left(e^{-x} g'(x)\right)' e^{-x}\, dx$$

$$= \int_0^\infty f^*(x) \left(e^{-x} g'(x)\right)' dx$$

$$\overset{\text{Ibp.}}{=} \underbrace{f^*(x)\, e^{-x} g'(x)\Big|_0^\infty}_{=0 \text{ because } f(0)=0,\ e^{-x}\to 0 \text{ as } x\to\infty} - \int_0^\infty \left(f'(x)\right)^* e^{-x} g'(x)\, d$$

$$\overset{\text{Ibp.}}{=} \underbrace{-\left(f'(x)\, e^{-x}\right)^* g(x)\Big|_0^\infty}_{=0} + \int_0^\infty \left(\left(f'(x)e^{-x}\right)'\right)^* g(x)\, dx$$

$$= \int_0^\infty \left(\mathcal{L}f\right)^*(x)\, g(x)\, dx = \langle \mathcal{L}f, g\rangle$$

□

5. Show that if f is an even, real-valued function, i.e. if $f(x) = f(-x)$ and $f^*(x) = f(x)$, then its Fourier transform is a real-valued function as well. (5)

6. (E) Show that $\mathcal{F}(f(x+a)) = e^{i\xi a}\mathcal{F}(f)$, where $\mathcal{F}(f) = \tilde{f}$ denotes the Fourier transform of f. (4)

(A) Show that if f is periodic with period a, then $\tilde{f}(\xi) = 0$ unless $\xi a = 2\pi n$ for some integer n. (5)

5. $$\left(\tilde{f}(\xi)\right)^* = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx\right)^*$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x} \underbrace{f^*(x)}_{=f(x)}\, dx \qquad z = -x \Rightarrow dx = -dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-i\xi z} \underbrace{f(-z)}_{=f(z)} (-dz)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi z} f(z)\, dz$$

$$= \tilde{f}(\xi)$$

$\Rightarrow \tilde{f}$ is real-valued.

6. (E)

$$\mathcal{F}(f(x+a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x+a)\, dx$$

$z = x+a$

$dx = dz$

$x = z-a$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{e^{-i\xi(z-a)}}_{= e^{-i\xi z} e^{i\xi a}} f(z)\, dz$$

$$= e^{i\xi a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi z} f(z)\, dz$$

$$= e^{i\xi a} \mathcal{F}(f) \qquad \square$$

(A) f is periodic with period a means that

$$f(x) = f(x+a)$$

Take Fourier transform:

$$\mathcal{F}(f) = \mathcal{F}(f(x+a))$$
$$= e^{i\xi a} \mathcal{F}(f) \qquad \text{from part (E)}$$

So either $\mathcal{F}(f)(\xi) = \tilde{f}(\xi) = 0$,

or $e^{i\xi a} = 1$

$\Rightarrow \xi a$ is an integer multiple of 2π.

$\square$

7. Compute

$$\int_{-\infty}^{\infty} \delta(e^{2x}-1)\, e^x\, dx.$$

(5)

Change variables:

$u = e^{2x} - 1 \quad \Rightarrow \quad e^{2x} = u+1 \quad \Rightarrow e^x = \sqrt{u+1}$

$\Rightarrow \frac{du}{dx} = 2e^{2x}$

$\Rightarrow dx = \frac{1}{2} e^{-2x}\, du$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(e^{2x}-1)\, e^x\, dx = \int_{-1}^{\infty} \delta(u)\, e^x \, \frac{1}{2} e^{-2x}\, du$$

$$= \frac{1}{2}\int_{-1}^{\infty} \delta(u)\, e^{-x}\, du$$

$$= \frac{1}{2}\int_{-1}^{\infty} \delta(u) \frac{1}{\sqrt{u+1}}\, du$$

$$= \frac{1}{2} \frac{1}{\sqrt{0+1}}$$

$$= \frac{1}{2}$$

8. (E) Show that the Fourier transform of

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-x} \sin x & \text{for } x \geq 0 \end{cases}.$$

is

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left(\frac{1}{1 - i + i\xi} - \frac{1}{1 + i + i\xi} \right).$$

(8)

(A) Prove that

$$\int_0^\infty e^{-2x} \sin^2 x \, dx = \frac{1}{\pi} \int_0^\infty \frac{1}{4 + \xi^4} \, d\xi. \qquad (*)$$

Hint: You may use the result from part (A) without proof. (10)

(E): $\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\xi x} e^{-x} \underbrace{\sin x}_{= \frac{1}{2i}(e^{ix} - e^{-ix})} dx$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \int_0^\infty \left(e^{(-i\xi - 1 + i)x} - e^{(-i\xi - 1 - i)x} \right) dx$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left(\frac{e^{(-i\xi-1+i)x}}{-i\xi - 1 + i} - \frac{e^{(-i\xi-1-i)x}}{-i\xi - 1 - i} \right) \Big|_0^\infty$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left(\frac{1}{i\xi + 1 - i} - \frac{1}{i\xi + 1 + i} \right) = \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \frac{2i}{(i\xi+1)^2 + 1}$

(A): $\int_0^\infty e^{-2x} \sin^2 x \, dx = \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(\xi)|^2 d\xi$ (Parseval)

$\overset{(E)}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi} \underbrace{\frac{1}{(1+i\xi)^2+1} \frac{1}{(1-i\xi)^2+1}}_{= \frac{1}{2-\xi^2+2i\xi} \frac{1}{2-\xi^2-2i\xi} = \frac{1}{(2-\xi^2)^2 + 4\xi^2} = \frac{1}{4+\xi^4}} d\xi$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{4 + \xi^4}$

$= \frac{1}{\pi} \int_0^\infty \frac{d\xi}{4 + \xi^4}$ $\square$