Engineering and Science Mathematics 2B

Midterm II

April 9, 2003

1. Diagonalize the matrix
$$A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} .$$
I.e., find a diagonal matrix D and a change of coordinate S such that $D = S^{-1}AS$.
(10)

2. (E) Are the eigenvectors of the matrix in Question 1 orthogonal?
Explain why you can answer this question without even computing the eigenvalues. (4)

(A) Let A be a Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Show that
$$A = \sum_{i=1}^{n} \lambda_i \, \boldsymbol{v}_i \, \boldsymbol{v}_i^H .$$
(5)

3. (E) Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by
$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \qquad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} .$$
(4)

(A) Find a basis of $\mathbb{R}^2$ which is orthonormal with respect to the non-standard inner product
$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T A \boldsymbol{v} \qquad \text{with } A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$
(5)

4. (E) Let A be a Hermitian matrix, i.e. $A = A^H$, and consider the standard inner product where $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^H \boldsymbol{v}$. Show that
$$\langle \boldsymbol{u}, A\boldsymbol{v} \rangle = \langle A\boldsymbol{u}, \boldsymbol{v} \rangle .$$
(4)

(A) Consider the vector space of bounded differentiable functions with bounded first derivatives which, moverover, satisfy $f(0) = 0$. On this vector space we define the inner product

$$\langle f, g \rangle = \int_0^\infty f^*(x)\, g(x)\, e^{-x}\, dx\,.$$

Show that the operator

$$\mathcal{L}f = e^x \frac{d}{dx}\left(e^{-x}\,\frac{df}{dx}\right)$$

is Hermitian, i.e. that $\langle f, \mathcal{L}g \rangle = \langle \mathcal{L}f, g \rangle$. (5)

5. Show that if f is an even, real-valued function, i.e. if $f(x) = f(-x)$ and $f^*(x) = f(x)$, then its Fourier transform is a real-valued function as well. (5)

6. (E) Show that $\mathcal{F}(f(x+a)) = e^{i\xi a}\,\mathcal{F}(f)$, where $\mathcal{F}(f) = \tilde{f}$ denotes the Fourier transform of f. (4)

   (A) Show that if f is periodic with period a, then $\tilde{f}(\xi) = 0$ unless $\xi a = 2\pi n$ for some integer n. (5)

7. Compute

$$\int_{-\infty}^{\infty} \delta(e^{2x} - 1)\, e^x\, dx\,.$$

(5)

8. (E) Show that the Fourier transform of

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-x}\,\sin x & \text{for } x \geq 0 \end{cases}.$$

is

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{2i}\left(\frac{1}{1-i+i\xi} - \frac{1}{1+i+i\xi}\right).$$

(8)

   (A) Prove that

$$\int_0^\infty e^{-2x}\,\sin^2 x\, dx = \frac{1}{\pi}\int_0^\infty \frac{1}{4+\xi^4}\, d\xi\,.$$

Hint: You may use the result from part (A) without proof. (10)