1. (a) Find the normal equation for the plane that passes through the points $(0,-1,1)$, $(2,2,-1)$, and $(1,1,3)$.

   (b) Find the distance of the point $(1,1,1)$ to this plane.

(5+5)

(a) Two vectors giving the orientation of the plane are

$$\vec{u} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

normal vector:

$$\vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix} 6-(-4) \\ -2-4 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ 1 \end{pmatrix}$$

normal equation:

$$(\vec{x} - \vec{a}) \cdot \vec{n} = 0 \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2+1 \\ x_3-1 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -6 \\ 1 \end{pmatrix} = 0$$

(b) $$\|\vec{n}\| = \sqrt{100+36+1} = \sqrt{137}$$

$$d = \left| \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right] \cdot \vec{n} \right|_2 \frac{1}{\|\vec{n}\|} = \frac{1}{\sqrt{137}} \left| \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -6 \\ 0 \end{pmatrix} \right|$$

$$= \frac{2}{\sqrt{137}}$$

2. Find the general solution to the system of linear equations $Ax = b$ with

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & -4 & 2 & -2 \\ 2 & -4 & 2 & 6 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$$

Check your answer! (10)

3. (E) Consider the matrix A from the previous question. Show that $Ax = c$ does not have a solution when

$$c = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(4)

   (A) Consider the matrix A from the previous question. Characterize all vectors $c \in \mathbb{R}^3$ such that $Ax = c$ is solvable. (5)

2.
$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 2 & -4 & 2 & -2 & -2 \\ 2 & -4 & 2 & 6 & 6 \end{array}\right) \xrightarrow[R2-2R1 \to R2]{R3-2R1 \to R3} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right)$$

$$\xrightarrow[R2+R3 \to R3]{-\frac{1}{4}R2 \to R2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \quad x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

Check:

$$A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} = b \quad \checkmark$$

$$A \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \checkmark$$

$$A \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \checkmark$$

3(E):

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 2 & -4 & 2 & -2 & 1 \\ 2 & -4 & 2 & 6 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 4 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right)$$

$\Rightarrow$ the system is inconsistent.

3(A): Solution (a): Find a basis for Range A; clearly, the first 3 columns are linearly dependent, so

$$V_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$$

are a basis for the Range. I.e. all vectors b of form

$$b = \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} \qquad (*)$$

will result in a consistent system.

Solution (b):

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & b_1 \\ 2 & -4 & 2 & -2 & b_2 \\ 2 & -4 & 2 & 6 & b_3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & b_1 \\ 0 & 0 & 0 & -4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 4 & b_3 - 2b_1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & b_1 \\ 0 & 0 & 0 & -4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 4b_1 \end{array}\right)$$

$\Rightarrow$ b must satisfy $b_2 + b_3 - 4b_1 = 0$.

Note: Eliminating the parameters in $(*)$ gives the same answer.

4. (E) Decide whether the following set of vectors is a basis of $\mathbb{R}^3$. Justify your answer.

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.$$

(4)

(A) Prove that the columns of any $n \times k$ matrix with $n < k$ are linearly dependent.

(5)

4(E): A basis of $\mathbb{R}^3$ consists of 3 vectors, and any linearly independent set of 3 vectors is a basis.

So we only need to check linear independence:

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ The vectors are l.d., hence not a basis.

(A): The column space (or range) of an $n \times k$ matrix is a subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$), and can therefore have at most n linearly independent vectors. Since there are $k > n$ column vectors, they must be l.d.

5. (E) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Write out a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$.

(c) Check your result by explicitly performing the matrix multiplications SD and AS.

(8+4+4)

(a)

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 - (1-\lambda)\cdot 3\cdot 3$$

$$= (1-\lambda)\left[1 - 2\lambda + \lambda^2 - 9\right]$$

$$= (1-\lambda)(\lambda-4)(\lambda+2)$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 4, \quad \lambda_3 = -2$$

For λ_1: $(A - \lambda_1 I)v_1 = 0$

$$\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

For λ_2: $(A - \lambda_2 I)v_2 = 0$

$$\begin{pmatrix} -3 & 0 & 3 \\ 0 & -3 & 0 \\ 3 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

For λ_3: $(A - \lambda_3 I)v_3 = 0$

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(b)

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

(c)

$$SD = \begin{pmatrix} 0 & -4 & -2 \\ -1 & 0 & 0 \\ 0 & -4 & 2 \end{pmatrix} = AS$$

(A) (a) Find the matrix A representing the linear transformation on $\mathbb{R}^2$ that maps the triangle with vertices $a = (0,0)$, $b = (1,0)$, $c = (1,1)$ onto the triangle with vertices $a' = (0,0)$, $b' = (\frac{1}{2}, \frac{3}{2})$, $c' = (2,2)$.

(b) Diagonalize A. I.e., find a diagonal matrix D and a change of coordinate S such that $D = S^{-1}AS$.

(c) Use the result from (b) to give a concise geometric description of the linear transformation.

(5+10+5)

(a) We need: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$

$$\Rightarrow a_{11} = \tfrac{1}{2}, \quad a_{21} = \tfrac{3}{2}$$

and $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\Rightarrow a_{11} + a_{12} = 2, \quad \text{so } a_{12} = 2 - \tfrac{1}{2} = \tfrac{3}{2}$$

$$a_{21} + a_{22} = 2, \quad \text{so } a_{22} = 2 - \tfrac{3}{2} = \tfrac{1}{2}$$

$$\Rightarrow A = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

(b) $\det(A - \lambda I) = (\frac{1}{2} - \lambda)^2 - \frac{9}{4} = \frac{1}{4} - \lambda + \lambda^2 - \frac{9}{4} = (\lambda+1)(\lambda-2)$

$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 2$

$(A - \lambda_1 I)v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $(A - \lambda_2 I)v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

(c) The transformation mirrors about the line $x_1 = x_2$, and stretches by a factor of 2 in the direction of the line.

6. Let V be the vector space of continuous functions on $\mathbb{R}$ spanned by the basis

$$E = \{e^{ix}, e^{-ix}\}$$

with the usual addition and scalar multiplication.

(a) Find the matrix D which represents the derivative operator on V with respect to the basis E.

(b) Let

$$T = \{\sin x, \cos x\}.$$

Explain why T is also a basis of V, and find the matrix S for the change of basis from T to E.
(Recall that $e^{ix} = \cos x + i \sin x$.)

(c) Find the matrix C which represents the derivative with respect to the new basis T.

(5+5+5)

(a) $\frac{d}{dx} e^{ix} = i e^{ix}$ $\qquad \frac{d}{dx} e^{-ix} = -i e^{-ix}$

$\Rightarrow D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

(b) We know that

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$ $\qquad (*)$

$\Rightarrow$ The basis E is in the span of T, and T has the same number of vectors

$\Rightarrow$ T is also a basis of V

Clearly from $(*)$ we see that $S^{-1} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

$\left(\begin{array}{cc|cc} i & -i & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & -1 & -i & 0 \\ 0 & 2 & i & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -\frac{i}{2} & \frac{1}{2} \\ 0 & 1 & \frac{i}{2} & \frac{1}{2} \end{array}\right)$

$\Rightarrow S = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$

(c) $C = S^{-1} D S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

7. (E) If $A \in M(n \times n)$, show that $B = A^T A$ is a symmetric matrix.
(Recall that a matrix C is symmetric if $C = C^T$.) (4)

(A) Prove that the eigenvalues of any real symmetric matrix are real. (5)

(E): $B^T = (A^T A)^T = A^T (A^T)^T = A^T A = B$

(A): Assume

$Av = \lambda v$ $\qquad$ for $v \neq 0$

and multiply from the left by v^H:

$v^H A v = \lambda v^H v$ $\qquad (*)$

Now take the Hermitian conjugate on both sides:

$v^H A^H ({}^H)^H = \lambda^* v^H (v^H)^H$

$\Rightarrow v^H A^H v = \lambda^* v^H v$ $\qquad (**)$

So if A is Hermitian (or real symmetric), the left sides of $(*)$ and $(**)$ are equal, hence

$\lambda v^H v = \lambda^* v^H v$

$\Rightarrow \lambda = \lambda^*$

so λ is real.