1. (a) Compute the inverse of the Matrix

$$A = \begin{pmatrix} i & 0 & -i \\ i & -i & 0 \\ 0 & i & i \end{pmatrix}.$$

(b) Using the result from part (a), or otherwise, solve the system of linear equations

$$Ax = \begin{pmatrix} i \\ 1 \\ -i \end{pmatrix}.$$

(10+5)

(a)
$$\left(\begin{array}{ccc|ccc} i & 0 & -i & 1 & 0 & 0 \\ i & -i & 0 & 0 & 1 & 0 \\ 0 & i & i & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -i & 0 & 0 \\ 1 & -1 & 0 & 0 & -i & 0 \\ 0 & 1 & 1 & 0 & 0 & -i \end{array}\right)$$

$$\xrightarrow{R2-R1\to R2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -i & 0 & 0 \\ 0 & -1 & 1 & i & -i & 0 \\ 0 & 1 & 1 & 0 & 0 & -i \end{array}\right) \xrightarrow[R3\to R2]{R2+R3\to R3} \quad \xrightarrow{\frac{1}{2}R3\to R3} \quad \ldots$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -i & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -i \\ 0 & 0 & 1 & \frac{i}{2} & -\frac{i}{2} & -\frac{i}{2} \end{array}\right) \xrightarrow[-R3+R2\to R2]{R3+R1\to R1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{i}{2} & -\frac{i}{2} & -\frac{i}{2} \\ 0 & 1 & 0 & -\frac{i}{2} & \frac{i}{2} & -\frac{i}{2} \\ 0 & 0 & 1 & \frac{i}{2} & -\frac{i}{2} & -\frac{i}{2} \end{array}\right)$$

$$\Rightarrow A^{-1} = \frac{i}{2}\begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

(b)
$$x = A^{-1}\begin{pmatrix} i \\ 1 \\ -i \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} i \\ 1 \\ -i \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -1 \\ 1 \\ -1+2i \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

2. (E) Find the matrix representing—with respect to the standard basis—the projection in $\mathbb{R}^2$ onto the vector

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(4)

(A) Let V be a real vector space with an inner product, $\{e_1, \ldots, e_n\}$ an orthonormal basis for V, and T a linear transformation on V. Explain why the matrix S which represents T with respect to the basis $\{e_1, \ldots, e_n\}$ has the components

$$s_{ij} = \langle T(e_j), e_i \rangle. \qquad (*)$$

(5)

(E) We need the unit vector in v-direction: $u = \frac{v}{\|v\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Long answer: Compute the result of projecting each standard basis vector an write resulting vectors into a matrix:

$$P_v(e_1) = \langle u, e_1 \rangle u = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$P_v(e_2) = \langle u, e_2 \rangle u = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ Matrix is $S = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Short answer: Projection of any vector x onto v is

$$P_v(x) = \langle u, x \rangle u = u u^T x = \underbrace{\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1\ -1)}_{= S = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} \cdot x$$

(A) For $v \in V$ write $v = v_1 e_1 + \ldots + v_n e_n$

Let $w = T(v)$ and write $w = w_1 e_1 + \ldots + w_n e_n$

S is the matrix such that $w_i = \sum_{j=1}^{n} s_{ij} v_j$

$\Rightarrow \sum_{j=1}^{n} s_{ij} v_j = \langle w, e_i \rangle = \langle T(v), e_i \rangle = \sum_{j=1}^{n} \langle T(e_j), e_i \rangle$

comparing coefficients yields $(*)$.

3. Consider the vector space P_2 of polynomials of degree less or equal to 2 with basis

$$B = \{1, x, x^2\}.$$

(a) What is the matrix S representing the derivative on P_2 with respect to the basis B?

(b) Find the eigenvalues and eigenvectors of S.

(c) Show that S is not diagonalizable. Is there another basis for P_2 such that the derivative with respect to this other basis is diagonalizable? Explain.

(5+5+5)

(a)
$$\left.\begin{aligned} &\frac{d}{dx}(1) = 0, \quad \text{coordinates are } \begin{pmatrix}0\\0\\0\end{pmatrix} \\ &\frac{d}{dx} x = 1, \quad \text{coordinates are } \begin{pmatrix}1\\0\\0\end{pmatrix} \\ &\frac{d}{dx} x^2 = 2x, \quad \text{coordinates are } \begin{pmatrix}0\\2\\0\end{pmatrix} \end{aligned}\right\} \Rightarrow S = \begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}$$

(b)
$$0 = \det(S-\lambda I) = \det\begin{pmatrix}-\lambda&1&0\\0&-\lambda&2\\0&0&-\lambda\end{pmatrix} = -\lambda^3$$

$\Rightarrow \lambda = 0$ is the only eigenvalue.

Corresponding eigenvector v must solve $\begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix} v = 0$

The solution is obviously (any multiple of) $v = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

(c) There is only one eigenvector, but we need 3 to form a basis

Diagonalizability is a property of the transformation independent of the chosen basis; the process of diagonalization will find the correct choice of basis which makes the transformation diagonal whenever this is possible.

So the answer is NO!

4. Compute the complex Fourier series for the function $f(x) = x$ on the interval $[-\pi, \pi]$. (10)

$$f_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ikx} \, x \, dx$$

If $k \neq 0$: Integrate by parts:

$$f_k = \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} e^{-ikx} \times \Big|_{-\pi}^{\pi} - \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} \underbrace{\int_{-\pi}^{\pi} e^{-ikx} \, dx}_{=0}$$

(2π-periodic integrand!)

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{k} \Big(\underbrace{e^{ik\pi}}_{=(-1)^k} \pi - \underbrace{e^{-ik\pi}}_{=(-1)^k} (-\pi) \Big)$$

$$= \sqrt{2\pi} \; i \, \frac{(-1)^k}{k}$$

If $k = 0$:
$$f_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x \, dx = 0$$ (odd integrand)

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \neq 0} \sqrt{2\pi} \; i \, \frac{(-1)^k}{k}$$

$$= i \sum_{k \neq 0} \frac{(-1)^k}{k}$$

5. (E) Show that

$$\mathcal{F}\left(\frac{df}{dx}\right) = i\xi\,\mathcal{F}(f)\,,$$

where $\mathcal{F}$ denotes the Fourier transform. (8)

(A) It is known that the Fourier transform of a Gaussian distribution is again a (non-normalized) Gaussian distribution. More precisely, if ϕ_σ denotes the Gaussian with mean zero and variance σ^2, i.e.

$$\phi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}\,,$$

then

$$\tilde{\phi}_\sigma(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\sigma^2\xi^2}{2}}\,.$$

Show that the convolution of two Gaussian distributions ϕ_σ and ϕ_μ is again a Gaussian. What is the variance of the resulting Gaussian? (10)

(E):

$$\mathcal{F}\left(\frac{df}{dx}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x}\, f'(x)\, dx$$

$$\overset{\text{I.b.p.}}{=} \frac{1}{\sqrt{2\pi}} \left[\underbrace{e^{-i\xi x} f(x) \Big|_{-\infty}^{\infty}}_{=0 \text{ provided } f(x)\to 0 \text{ as } x\to\pm\infty.} - \int_{-\infty}^{\infty} (-i\xi)\, e^{-i\xi x} f(x)\, dx \right]$$

$$= i\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx = i\xi\, \mathcal{F}(f)$$

(A):

$$\mathcal{F}(\phi_\sigma * \phi_\mu) = \sqrt{2\pi}\; \tilde{\phi}_\sigma \cdot \tilde{\phi}_\mu = \sqrt{2\pi}\, \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2\xi^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2\xi^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(\sigma^2+\mu^2)\xi^2}{2}}$$

We recognize this as the Fourier transform of a standard Gaussian with variance $\sigma^2+\mu^2$.

6. (E) Compute

$$\int_{-\infty}^{\infty} \delta(x^{1/3})\, dx\,.$$

(8)

(A) If X is a continuous random variable with probability density function $p(x)$, and $Y = g(X)$ another random variable, then the probability density function of Y is given by

$$q(y) = \int_{-\infty}^{\infty} p(x)\, \delta(g(x) - y)\, dx\,.$$

Compute $q(y)$ when $g(x) = x^{1/3}$, and $p(x) = 1$ on the interval $[0, 1]$ and $p(x) = 0$ otherwise.

(10)

(E): Set $u = x^{\frac{1}{3}} \Rightarrow x = u^3 \Rightarrow dx = 3u^2\, du$

Then $\displaystyle\int_{-\infty}^{\infty} \delta(x^{\frac{1}{3}})\, dx = \int_{-\infty}^{\infty} \delta(u)\, 3u^2\, du = 3u^2\Big|_{u=0} = 0$

(A): $\displaystyle q(y) = \int_0^1 1 \cdot \delta(x^{\frac{1}{3}} - y)\, dx$

Set $u = x^{\frac{1}{3}} - y \Rightarrow x = (u+y)^3 \Rightarrow dx = 3(u+y)^2\, du$

$\displaystyle\Rightarrow q(y) = \int_{-y}^{1-y} \delta(u)\, 3(u+y)^2\, du$

- So if $-y < 0 < 1-y$, i.e. if $0 < y < 1$, then

  $q(y) = 3(u+y)^2\Big|_{u=0} = 3y^2$.

- If $-y > 0$ or $1-y < 0$, i.e. if $y < 0$ or $y > 1$, then

  $q(y) = 0$

- If $y = 0$ or $y = 1$, then q is undefined (that's OK).

7. Each question on a multiple choice exam has three possible answers, of which only one is correct. Some student did not study very hard and there is only a 50% chance that he knows the anwer to a particular question. Of course he will select an answer at random in case he does not know the correct one. What is the probability that he knows the answer to a question he has anwered correctly? (10)

Let K be the event that the student knows the answer

C " " " that the student answers correctly.

A decomposition of the sample space gives

$$P(C) = \underbrace{P(C|K)}_{=1}\ \underbrace{P(K)}_{=\frac{1}{2}} + \underbrace{P(C|\bar{K})}_{=\frac{1}{3}}\ \underbrace{P(\bar{K})}_{=1-P(K)=\frac{1}{2}}$$

$$= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

Now use Bayes' Rule:

$$P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{P(C|K)\,P(K)}{P(C)}$$

$$= \frac{1 \cdot \frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$$

8. A Homework set in ESM 2B has six questions. For the corresponding quiz, three of these questions are selected at random, each selection being independent of the others.

(E) What is the probability that you and your best friend get an identical quiz? (8)

(A) Find a formula for the probability that in a group of k students at least two get an identical quiz. (10)

(E) The number of different quizzes (order of questions does not matter) is ${}^6C_3 = \frac{6!}{3!\,3!} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$

So the probability of a second person getting the quiz that a first person has got is $\frac{1}{20}$.

(A) P(at least 2 students have identical quizzes)

$= 1 - \underbrace{P(\text{all quizzes are different})}$

$= \dfrac{\#\text{ of possibilities to distribute } k \text{ distinct quizzes}}{\#\text{ of possibilities to distribute } k \text{ quizzes}}$

- The number of ways to distribute k distinct quizzes is zero if $k > n$, and ${}^nP_k$ if $0 \le k \le n$, where $n = 20$ is the number of distinct quizzes.
- The number of ways to distribut k possibly identical quizzes is n^k.

$\Rightarrow$ P(at least 2 students have identical quizzes) $= 1 - \dfrac{{}^nP_k}{n^k}$.

9. An experiment is independently repeated n times. The experiment has two possible outcomes: "success" with probability p and "failure" with probability $q = 1 - p$. Consider the random variable

$$X = \text{number of trials required to obtain the first success}.$$

(a) Explain why the probability function of X is

$$P(X = k) = p\,q^{k-1}.$$

(b) Show that the moment generating function $M_X(t) = E[e^{tX}]$ is given by

$$M_X(t) = \frac{p\,e^t}{1 - q\,e^t}.$$

(c) Compute, by using the moment generating function, or otherwise, the mean and the variance of X.

(5+5+5)

(a) The probability of one success after $k-1$ failures, the trials being statistically independent, is the product of the individual probabilities:

$$P(X=k) = \underbrace{q \cdot \ldots \cdot q}_{(k-1)\text{ failures}} \cdot \underbrace{p}_{1\text{ success}} = p\,q^{k-1}$$

(b) $$M_X(t) = E[e^{tX}] = \sum_{k=1}^{\infty} e^{tk} P(X=k) = p \sum_{k=1}^{\infty} e^{tk} q^{k-1}$$

$$= pe^t \sum_{k=1}^{\infty} (e^t q)^{k-1} = \frac{pe^t}{1-qe^t} = \frac{p}{e^{-t}-q}$$

(c) $$M_X'(t) = p\,\frac{e^{-t}}{(e^{-t}-q)^2}$$

$$M_X''(t) = p\,\frac{-e^{-t}(e^{-t}-q)^2 - (-e^{-t})\,2(e^{-t}-q)}{(e^{-t}-q)^4}$$

$$\Rightarrow E[X] = M_X'(0)$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$V[X] = E[X^2] - E[X]^2$$

$$= M_X''(0) - E[X]^2$$

$$= p\,\frac{-(1-q)^2 + 2(1-q)}{(1-q)^4} - \left(\frac{1}{p}\right)^2$$

$$= p\,\frac{-p^2 + 2p}{p^4} - \frac{1}{p^2}$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$= \frac{q}{p^2}$$