

# The Matrix Inverse

Marcel Oliver

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## 1 Properties of the Matrix Inverse

Given a matrix  $A \in M(n \times n)$ , its inverse  $A^{-1}$  is the unique matrix with the property

$$AA^{-1} = I = A^{-1}A.$$

Note the following identities (see Riley, Hobson & Bence, p. 271):

(i)  $(A^{-1})^{-1} = A$

(ii)  $(A^T)^{-1} = (A^{-1})^T$

(iii)  $(A^H)^{-1} = (A^{-1})^H$

(iv)  $(AB)^{-1} = B^{-1}A^{-1}$

Moreover, if  $A$  is invertible then the solution to the system of linear equations  $A\mathbf{x} = \mathbf{b}$  can be written

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

This observation tells us how we can compute the matrix inverse once we know how to solve linear equations. We begin by writing  $\mathbf{b}$  as a linear combination of the standard unit vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$ ,

$$\mathbf{b} = b_1\mathbf{e}_1 + \dots + b_n\mathbf{e}_n.$$

Then

$$\begin{aligned} \mathbf{x} &= A^{-1}(b_1\mathbf{e}_1 + \dots + b_n\mathbf{e}_n) \\ &= b_1A^{-1}\mathbf{e}_1 + \dots + b_nA^{-1}\mathbf{e}_n \\ &= \begin{pmatrix} | & & | \\ A^{-1}\mathbf{e}_1 & \dots & A^{-1}\mathbf{e}_n \\ | & & | \end{pmatrix} \mathbf{b}. \end{aligned}$$

Notice that the vectors  $\mathbf{x}_1 = A^{-1}\mathbf{e}_1, \dots, \mathbf{x}_n = A^{-1}\mathbf{e}_n$  are the columns of  $A^{-1}$ . At the same time, we see that these vectors are the solutions to the  $n$  linear equations

$$A\mathbf{x}_1 = \mathbf{e}_1, \quad \dots, \quad A\mathbf{x}_n = \mathbf{e}_n.$$

To find  $A^{-1}$ , we therefore have to simultaneously solve  $n$  inhomogeneous linear equations, which is the essence of the following procedure.

## 2 Computing the inverse

### Form the augmented matrix

Write the matrix  $A$  to the left, and the identity matrix to the right. For example, when

$$A = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix},$$

write

$$M = \left( \begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array} \right).$$

### Reduce to row-echelon form

Use elementary row transformations on the augmented matrix to bring the left-hand matrix into row echelon form.

- If you can obtain the identity matrix in the left-hand block, the matrix is invertible and the final right-hand block is  $A^{-1}$ .
- If you obtain a row of zeros in the left-hand block, i.e. if  $\text{rank } A < n$ , then  $A$  is not invertible.

Let's work out the example:

$$\begin{aligned} & \left( \begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\text{reorder rows}} \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array} \right) \xrightarrow{\text{R2}-2\text{R1}\rightarrow\text{R2}} \\ & \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array} \right) \xrightarrow{\text{R3}-\text{R2}\rightarrow\text{R3}} \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \\ 0 & 0 & \frac{1}{2} & 1 & 2 & -1 \end{array} \right) \xrightarrow{\begin{array}{l} 2\text{R2}\rightarrow\text{R2} \\ 2\text{R3}\rightarrow\text{R3} \end{array}} \\ & \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -4 & 2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array} \right) \xrightarrow{\begin{array}{l} \text{R1}-\text{R3}\rightarrow\text{R1} \\ \text{R2}+2\text{R3}\rightarrow\text{R2} \end{array}} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array} \right) \end{aligned}$$

### Check your solution

We see that

$$A^{-1} = \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix}$$

and it is easy to check that

$$AA^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$