Gaussian elimination: How to solve systems of linear equations

Marcel Oliver

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Step 1: Write out the *augmented matrix*

A system of linear equation is generally of the form

$$A\,\boldsymbol{x} = \boldsymbol{b}\,, \tag{1}$$

where $A \in M(n \times m)$ and $\boldsymbol{b} \in \mathbb{R}^n$ are given, and $\boldsymbol{x} = (x_1, \ldots, x_m)^T$ is the vector of unknowns. For example, the system

$$\begin{aligned} x_2 + 2\,x_3 - x_4 &= 1 \\ x_1 + x_3 + x_4 &= 4 \\ -x_1 + x_2 - x_4 &= 2 \\ 2\,x_2 + 3\,x_3 - x_4 &= 7 \end{aligned}$$

can be written in the form (1) with

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix}\,, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 7 \end{pmatrix}\,.$$

To simplify notation, we write A and $\boldsymbol{b}$ into a single *augmented matrix*,

$$M = \left(\begin{array}{cccc|c} 0 & 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 & 4 \\ -1 & 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right)\,. \tag{2}$$

Step 2: Bring M into *reduced row echelon form*

The goal of this step is to bring the augmented matrix into *reduced row echelon form*. A matrix is in this form if

- the first non-zero entry of each row is 1, this element is referred to as the *pivot*,
- each pivot is the only non-zero entry in its column,
- each row has at least as many leading zeros as the previous row.

For example, the following matrix is in row echelon form, where $*$ could be any, possibly non-zero, number:

$$\begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Three types of *elementary row operations* are permitted in this process, namely

(A) exchanging two rows of M,

(B) multiplying a row by a non-zero scalar,

(C) adding a multiple of one row to another row.

As an example, we row-reduce the augmented matrix (2):

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 & 4 \\ -1 & 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right) \xrightarrow{\text{reorder rows}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ -1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right) \xrightarrow{\text{R1+R2}\to\text{R2}}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right) \xrightarrow[\text{R4}-2\,\text{R2}\to\text{R4}]{\text{R3}-\text{R2}\to\text{R3}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & -1 & -5 \end{array}\right) \xrightarrow{\text{R4}-\text{R3}\to\text{R4}}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{R2}-\text{R3}\to\text{R2}]{\text{R1}-\text{R3}\to\text{R1}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 9 \\ 0 & 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Step 3: Zero, one, or many solutions?

There are two fundamentally different situations:

- The matrix A is *regular*. In this case, the left-hand block of M has been reduced to the identity matrix. There is exactly one solution, independent of which vector $\boldsymbol{b}$ you started out with.

- The matrix A is *degenerate*. In this case, the left-hand block of the row-reduced augmented matrix has more columns than non-zero rows. Then, dependent on which vector $\boldsymbol{b}$ you started out with, there is either no solution at all (the system is *inconsistent*), or an infinite number of solutions (the system is *underdetermined*).

If the rightmost column of the row-reduced augmented matrix has a nonzero entry in a row that is otherwise zero, the system is inconsistent.

Otherwise, the general solution has the following structure. It is the sum of a *particular solution* of the *inhomogeneous equation* $A\boldsymbol{x} = \boldsymbol{b}$ and the *general solution* of the *homogeneous equation* $A\boldsymbol{x} = 0$.

Step 4: Write out the solution

- If the left-hand block of the row-reduced matrix is not square, make it square by adding or removing rows of zeros. This has to be done in such a way that the leading 1 in each row (the *pivot*) lies on the diagonal!

- The rightmost column of the row-reduced augmented matrix is a particular solution.

- To find a basis for the general solution of the homogeneous system, proceed as follows: Take every column of the row-reduced augmented matrix that has a zero on the diagonal. Replace that zero by -1. The set of these column vectors is the basis you need.

In the example above, a particular solution is $(9, 11, -5, 0)^T$ and the general solution of the homogeneous equation is a one-dimensional subspace with basis vector $(2, 1, -1, -1)^T$. Therefore, the general solution to the inhomogeneous equation $A\boldsymbol{x} = \boldsymbol{b}$ is the line

$$\boldsymbol{x} = \begin{pmatrix} 9 \\ 11 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Another example: Assume that the row-reduced matrix is

$$\left(\begin{array}{cccccc|c} 0 & 0 & 1 & -3 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 1 & 6 & 7 \end{array}\right) .$$

Padding the matrix with the required rows of zeros gives

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) ,$$

and the general solution is

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 7 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ -3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 6 \\ -1 \end{pmatrix} .$$

Step 5: Check your solution

By multiplying A with the vectors representing the solution, you can easily verify that the computation is correct. In our example,

$$A \begin{pmatrix} 9 \\ 11 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ 11 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 7 \end{pmatrix} = \boldsymbol{b} ,$$

$$A \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$