1. Prove by induction that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

for every $n \in \mathbb{N}$. (8)

$\underline{n=1:}$ There is only one term in the sum on the left, namely

$$\frac{1}{1\cdot 2} = \frac{n}{n+1} \quad \text{for } n=1 .$$

$\underline{n \to n+1:}$

$$\frac{1}{1\cdot 2} + \dots + \frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)(n+2)}$$

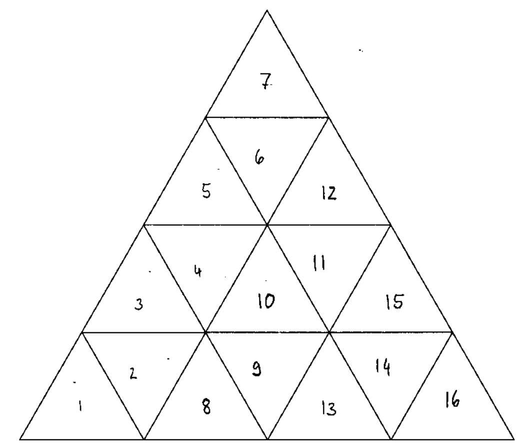
$$\overset{\text{I.H.}}{=} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)}$$

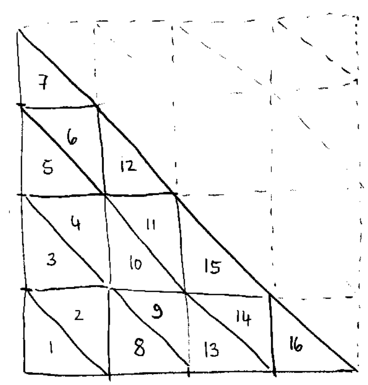
$$= \frac{(n+1)}{(n+1)+1} \qquad \square$$

2. A triangle is divided by lines parallel to each of its sides into T_n smaller triangles. (The figure below shows the case $n=4$ where $T_n = 16$.) Find a formula for T_n and prove that your formula is correct. (8)



$\underline{\text{Solution 1:}}$

Note that this figure can be distorted without changing the number of small triangles:



The dashed lines show a mirror image of the triangle.

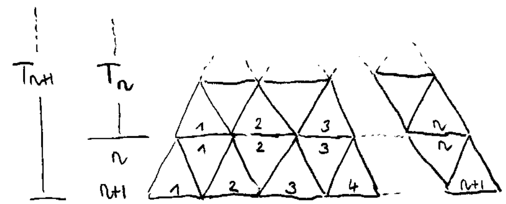
We see that the number of small triangles (unmirrored) equals the number of squares on a 4x4 grid.

Thus, we obtain

$$T_n = n^2$$

Solution 2:

Note that the T_n satisfy the following recurrence relation:



$$\Rightarrow T_{n+1} = T_n + n + (n+1) = T_n + 2n + 1$$

Claim: $T_n = n^2$

Proof by induction:

$n=1$: $T_1 = 1^2 = 1$ (a single triangle)

$n \to n+1$:
$$\begin{aligned} T_{n+1} &= T_n + 2n + 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$
$\square$

3. Give one example each of a function $f\colon \mathbb{N} \to \mathbb{N}$ that is

   (a) bijective,
   (b) injective but not surjective,
   (c) surjective but not injective,
   (d) neither surjective nor injective.

(2+2+2+2)

(a) The identity map $f(n) = n$

(both injectivity and surjectivity are trivial)

OR: $f(n) = \begin{cases} n+1 & \text{if } n \text{ odd} \\ n-1 & \text{if } n \text{ even} \end{cases}$

(flips even and odd numbers pair-wise)

(b) The map s from the Peano axioms

(injective by assumption, not surjective as $1 \notin s(\mathbb{N})$.)

(c) $f(n) = \begin{cases} n-1 & \text{if } n \neq 1 \\ 1 & \text{if } n = 1 \end{cases}$

(not injective as $f(1) = f(2) = 1$, surjective as each $n \in \mathbb{N}$ has $s(n)$ as one of its preimages)

(d) $f(n) = 1$ for every $n \in \mathbb{N}$

(not injective as $f(1) = f(2) = 1$, not surjective as $2 \notin f(\mathbb{N})$.)

4. Consider the set $Q = \{(a, b) : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. Let $(a, b) \sim (c, d)$ if and only if $ad = bc$.

(a) Show that $\sim$ is an equivalence relation, i.e. that it is reflexive, symmetric, and transitive.

(b) Define an operation $\circ$ on Q via $(a, b) \circ (c, d) = (ad + bc, bd)$. Show that $\circ$ is well defined on classes $[a, b]$ with respect to the equivalence relation $\sim$. In other words, prove that if $(a, b) \sim (a', b')$, then $(a, b) \circ (c, d) \sim (a', b') \circ (c, d)$.

(6+4)

(a) • $ab = ba \Rightarrow (a,b) \sim (a,b)$ (reflexivity)

• $ad = bc \Rightarrow cb = da$

$\Updownarrow \qquad \Updownarrow$

$(a,b) \sim (c,d) \qquad (c,d) \sim (a,b)$ (symmetry)

• $(a,b) \sim (c,d) \Rightarrow ad = bc \Rightarrow adf = bcf$
$(c,d) \sim (e,f) \Rightarrow cf = de \Rightarrow bcf = bde$
$\Rightarrow adf = bde \overset{d \neq 0}{\Downarrow} af = be$

$(a,b) \sim (e,f) \Leftarrow af = be$

(reflexivity)

(b) $(a,b) \sim (a',b') \Rightarrow ab' = ba'$ $(*)$

$(a,b) \circ (c,d) = (ad + bc, bd)$

$(a',b') \circ (c,d) = (a'd + b'c, b'd)$

$(ad + bc)(b'd) = adb'd + bcb'd \overset{(*)}{=} a'dbd + b'cbd$

$= (a'd + b'c)bd$

$\Rightarrow (a,b) \circ (c,d) \sim (a',b') \circ (c,d)$

5. Consider a map $G \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following properties:

(M1) $G(a, 1) = a$ for all $a \in \mathbb{N}$,

(M2) $G(a, s(b)) = G(a, b) + a$ for all $a, b \in \mathbb{N}$.

Prove, by showing that a certain set is inductive, that $G(a, c) = G(b, c)$ implies $a = b$ for any $a, b, c \in \mathbb{N}$.

Let $M = \{c \in \mathbb{N} : G(a,c) = G(b,c) \text{ implies } a = b \text{ for } a, b \in \mathbb{N}\}$

$1 \in M$: $G(a,1) = G(b,1) \overset{(M1)}{\Rightarrow} a = b$

$s(M) \subseteq M$: Let $c \in M$.

$G(a, s(c)) = G(b, s(c))$

$\overset{(M2)}{\Rightarrow} G(a,c) + a = G(b,c) + b$ $(**)$

$\overset{(*)}{\Rightarrow} a = b$

$\Rightarrow s(c) \in M$.

Step $(*)$ is not obvious ! To prove it, we use the following

Lemma: If $a < b$ for $a, b \in \mathbb{N}$, then $G(a,c) < G(b,c)$ for any $c \in \mathbb{N}$.

Proof: Let $M = \{c \in \mathbb{N} : a < b \Rightarrow G(a,c) < G(b,c)\}$

$1 \in M$: $a < b \overset{(M1)}{\Rightarrow} G(a,1) < G(b,1)$

$s(M) \subseteq M$: Let $c \in M$. Then

$G(a, s(c)) \overset{(M2)}{=} G(a,c) + a < G(a,c) + b \overset{c \in M}{<} G(b,c) + b = G(b, s(c))$

$\square$

Now suppose, WLOG, that $a < b$. Then $G(a,c) < G(b,c) \Rightarrow G(a,c) + a < G(b,c) + b$.

This contradicts $(*)$, so $a = b$ must be true.

6. Show that $\mathbb{N} \cong \{n \in \mathbb{N} : n \text{ even}\}$. (8)

Let $\phi : \mathbb{N} \to \{n \in \mathbb{N} : n \text{ even}\}$

$n \mapsto 2n$

We have to show:

(a) ϕ injective:

Assume $\phi(n) = \phi(m)$

$\Rightarrow 2n = 2m$

$\Rightarrow n = m$ (by Problem 5, for example)

(b) ϕ surjective:

By definition of the even numbers as

$\{n \in \mathbb{N} : n \text{ even}\} = \{2k : k \in \mathbb{N}\}$