General Mathematics and Computational Science I

Final Exam

December 20, 2006

1. Show that the binary operation on $\mathbb{Z}$ defined through

$$a \circ b = a + b - ab$$

is associative, i.e. that $a \circ (b \circ c) = (a \circ b) \circ c$. (10)

Solution:

$a \circ (b \circ c) = a + (b \circ c) - a(b \circ c) = a + (b + c - bc) - a(b + c - bc) = a + b + c - -ab - bc - ca + abc$

$(a \circ b) \circ c = (a \circ b) + c - (a \circ b)c = (a + b - ab) + c - (a + b - ab)c = a + b + c - -ab - bc - ca + abc$

It follows that $a \circ (b \circ c) = (a \circ b) \circ c$, i.e. the binary operation $\circ$ is associative.

2. Let $f \colon \mathbb{N} \to \mathbb{N}$ be a function with the property that $f(m + n) = f(m) + f(n)$ for all $m, n \in \mathbb{N}$. Find a formula for f and prove it by induction. (10)

Solution:

Notice first that for $m = 1$, we have $f(n+1) = f(n) + f(1)$ for all $n \in \mathbb{N}$. Let $f(1) = c$ be a constant in $\mathbb{N}$. Then, we observe that $f(2) = f(1) + f(1) = 2c$, $f(3) = 3c$ and claim that $f(n) = nc$ for all $n \in \mathbb{N}$.

Indeed, suppose that $f(n) = nc$ for some $n \in \mathbb{N}$. Then $f(n + 1) = f(n) + f(1) = nc + c = (n + 1)c$, which proves the claim by induction.

3. Which of the following relations is an equivalence relation, i.e. is reflexive, symmetric, and transitive? Give explicit proofs when a relation is an equivalence relation and a counter example when it is not.

   (a) On $\mathbb{Z}$, let $a \sim b$ if and only if $a \leq b$.

   (b) On $\mathbb{Z}$, let $a \sim b$ if and only if $|a - b| \leq 10$.

   (c) Let X be a set and $U \subset X$. For any $A, B \subset X$, let $A \sim B$ if and only if $A \cap U = B \cap U$.

(5+5+5)

Solution:

(a) This is not an equivalence relation, because it is not symmetric. For example, $0 \sim 1$ but $1 \nsim 0$.

(b) This is not an equivalence relation, because it is not transitive. We have $10 \sim 0$ and $0 \sim -10$ but $10 \nsim -10$, because $|10-(-10)| = 20 \nleq 10$.

(c) We check that this is an equivalence relation. $A \sim A \Leftrightarrow A \cap U = A \cap U$, hence the relation is reflexive. If $A \sim B \Leftrightarrow A \cap U = B \cap U$, then also $B \sim A$, hence the relation is symmetric. If $A \sim B$ and $B \sim C$, then $A \cap U = B \cap U = C \cap U$ hence $A \sim C$ and the relation is transitive.

4. Show that
$$(a\,\sin\theta + b\,\cos\theta)^2 \leq a^2 + b^2\,.$$
(10)

Solution 1:

By the Cauchy Schwarz inequality, $(a^2+b^2)(\sin^2\theta + \cos^2\theta) \geq (a\,\sin\theta + b\,\cos\theta)^2$, which proves the inequality above since $\sin^2\theta + \cos^2\theta = 1$.

Solution 2:

$(a\,\sin\theta + b\,\cos\theta)^2 = (a^2\,\sin^2\theta + b^2\,\cos^2\theta + 2ab\,\sin\theta\,\cos\theta) \leq a^2 + b^2$. This is equivalent to $2ab\,\sin\theta\,\cos\theta \leq a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) = a^2\cos^2\theta + b^2\sin^2\theta$. However, $2ab\,\sin\theta\,\cos\theta \leq a^2\cos^2\theta + b^2\sin^2\theta \Leftrightarrow 0 \leq (a\,\cos\theta - b\,\sin\theta)^2$, which is obvious.

5. A coin is tossed four times. Is it more likely to come up (a) exactly twice with the same face or (b) exactly three times with the same face? (10)

Solution 1:

The event of coming up (a) exactly twice with the same face can only be realized by two heads and two tails. The number of possible sequences of this kind is $\binom{4}{2} = 6$.

The event of coming up (b) with the same fact exactly three times can be realized by three heads and one tail, or vice versa. There are $\binom{4}{3} = 4$ possible sequences of each kind, so 8 possibilities altogether.

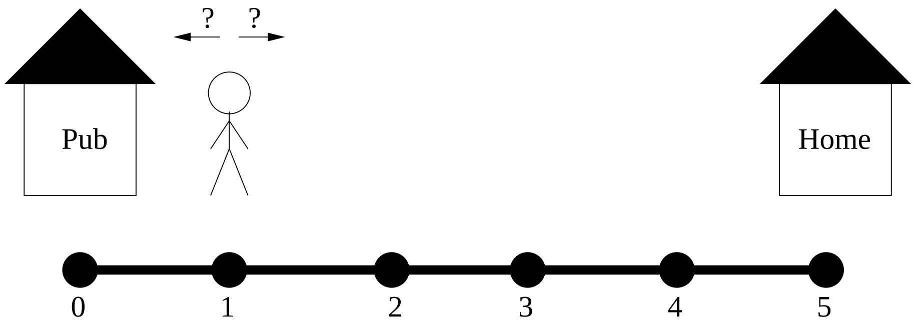
Hence, (b) is more likely to occur than (a).

Solution 2:

A face of a coin can be either head (H) or tail (T), which occur with equal probability, $\frac{1}{2}$. If the coin is tossed four times then the following situations occur: HHHH, HHHT, HHTH, HHTT, HTHH, HTTH, HTTT, HTHT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH and TTTT. There are a total of 16 possible cases. By brute-force counting, there are 6 and 8 possibilities that (a), respectively (b) happens. Hence the probability to obtain (a) is $\frac{6}{16}$ and the probability to obtain (b) is $\frac{8}{16}$. In conclusion, it is more likely to come up with (b).

6. A drunkard lives five houses up the street from the pub. He has just left the pub in the direction of home, but has lost orientation and will move one house down or one house up with probability $\frac{1}{2}$ each. If he gets home, he will stay there. If he gets back into the pub, his friends will buy him another drink and he will stay in the pub till the next morning. What is the probability that he gets home?

   The picture shows the drunkard about to start his way home.

   Hint: The probability p_n that he finds home from house no. n satisfies the difference equation $p_n = \frac{1}{2}\, p_{n-1} + \frac{1}{2}\, p_{n+1}$. Explain why. What are the boundary conditions? Then solve this difference equation. (10)

   Solution 1:

   Let 0 denote the pub and 1,2,3,4,5 denote the five houses up the street from the pub, where 5 is home. We need to compute p_1, the probability to go home from the starting position drawn in the picture. Clearly $p_5 = 1$ and $p_0 = 0$. If the drunkard is at house n, then with probability $\frac{1}{2}$ he moves in the direction home and with probability $\frac{1}{2}$ in the other direction (pub). Therefore we have $p_n = \frac{1}{2}\, p_{n-1} + \frac{1}{2}\, p_{n+1}$, for $n = 1, 2, 3, 4$.

   This difference equation has the characteristic polynomial $p(x) = \frac{1}{2}\, x^2 - x + \frac{1}{2} = \frac{1}{2}\,(x-1)^2$. Thus, it has a double root 1, so that the general solution is of the form $p_n = A + B\,n$. $p_0 = 0$ implies $A = 0$ and $p_5 = 1$ then implies that $B = \frac{1}{5}$. Thus, $p_n = n/5$ so that, in particular, $p_1 = \frac{1}{5}$.

   Solution 2:

   The difference equation can also be solved by elementary means. Note that, for $n = 1$, we have $p_1 = \frac{1}{2}\, p_2$.

   By iterating the recurrence, we find that $p_2 = \frac{1}{2}\, p_1 + \frac{1}{2}\, p_3$ so that $2\, p_1 = \frac{1}{2}\, p_1 + \frac{1}{2}\, p_3$ and finally $3\, p_1 = p_3$.

   Similarly, $p_3 = \frac{1}{2}\, p_2 + \frac{1}{2}\, p_4 = p_1 + \frac{1}{2}\, p_4$ so that $4\, p_1 = p_4$.

   Finally, $p_4 = \frac{1}{2}\, p_3 + \frac{1}{2}$, since $p_5 = 1$, so that $4\, p_1 = \frac{3}{2}\, p_1 + \frac{1}{2}$ or $p_1 = \frac{1}{5}$.

7. Consider the difference equation

$$x_{n+1} = x_n^2 - c$$

   where c is a real number.

(a) Find all equilibrium points. For which values of c do equilibrium points exist?

(b) Determine the stability of the equilibrium points as a function of c.

(It is sufficient to use the derivative criterion for stability even though it is inconclusive for certain values of c.)

(10+5)

Solution:

(a) Equilibrium points are given by the real solutions to the equation $x^2 - x - c = 0$. They exist when $1 + 4c \geq 0 \Leftrightarrow c \geq \frac{-1}{4}$. In this case, the equilibrium points are $x = \frac{1 \pm \sqrt{1+4c}}{2}$.

(b) We have $x_{n+1} = x_n^2 - c = f(x_n)$, where $f(x) = x^2 - c$. An equilibrium point, say x, is stable when $|f'(x)| < 1$ and unstable when $|f'(x)| > 1$. In our case $f'(x) = 2x$, hence the equilibrium point x is stable when $2|x| < 1$ and unstable when $2|x| > 1$. If $x = \frac{1+\sqrt{1+4c}}{2}$, then $2|x| > 1$ and x is always unstable. If $x = \frac{1-\sqrt{1+4c}}{2}$ then x is stable iff $2|x| < 1 \Leftrightarrow -1 < 1 - \sqrt{1+4c} < 1 \Leftrightarrow 0 < \sqrt{1+4c} < 2$. Hence x is stable whenever $\frac{-1}{4} < c < \frac{3}{4}$.
$x = \frac{1-\sqrt{1+4c}}{2}$ is unstable iff $2|x| > 1$. In this case, x is unstable iff $2x < -1$, i.e. $1 - \sqrt{1+4c} < -1 \Leftrightarrow 2 < \sqrt{1+4c} \Leftrightarrow \frac{3}{4} < c$.

8. Prove that

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} 2^{n-k} = \frac{3^n - 1}{2}.$$

Hint: Binomial Theorem. (10)

Solution:

By the Binomial Theorem we have $(2+1)^n = \sum_{k=1}^{n} \binom{n}{k} 2^{n-k}$ and $(2-1)^n = \sum_{k=1}^{n} \binom{n}{k} 2^{n-k}(-1)^k$. If we subtract the two relations we get:

$$3^n - 1 = 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} 2^{n-k}$$

.

and the equality to prove follows.

9. Consider n-words, i.e. words of length n, from the alphabet $\{A, B, C\}$.

(a) Count the number of different n-words.

(b) Count the number of different n-words with an odd number of As.

Hint: Use the result of Question 8.

(5+10)

Solution 1:

(a) Since each letter of an n-word can be chosen in exactly 3 ways, the number of different n-words is 3^n.

(b) Consider the number of n-words that contain the letter A exactly k times. Among these words, there are $\binom{n}{k}$ possibilities to distribute the As. The remaining $n-k$ positions are then filled from the alphabet $\{B, C\}$, there are 2^{n-k} possibilities for doing so. Thus, the number of n words that contain the letter A exactly k times is $\binom{n}{k} 2^{n-k}$. Thus, referring to Question 8, there are

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} 2^{n-k} = \frac{3^n - 1}{2}$$

possibilities to have n-words with an odd number of As.

Solution 2:

(b) Let y_n and x_n denote the number of different n-words with an odd, respectively an even number of As. Suppose now that we have an n-word with odd number of As. Then from this one we can form an $(n+1)$-word with an odd number of As by adding letters B or C only. Suppose that we have an n-word with an even number of As, then we can obtain an $(n+1)$-word by adding letter A only. Thus $y_{n+1} = 2y_n + x_n$. Since $y_n + x_n = 3^n$; it follows that $y_{n+1} = 3^n + y_n$. Summing the first n relations of this kind, we get that $y_n = \sum_{k=1}^{n-1} 3^k + y_1$, where $y_1 = 1$. Hence $y_n = \frac{3^n-1}{2}$.

10. Let x_n denote the number of n-words from the alphabet $\{A, B, C\}$ with an even number of As and let y_n denote the number of such n-words with an odd number of As.

(a) Find a recurrence relation which expresses x_{n+1} and y_{n+1} in terms of x_n and y_n.

(b) Rewrite this recurrence relation as a linear second-order difference equation in y_n. What are the starting values?

Hint: You should find that

$$y_{n+2} - 4\, y_{n+1} + 3\, y_n = 0\,.$$

(c) Solve this difference equation.

(5+5+5)

Solution:

(a) In Solution 2 to Problem 9 we already argued that $y_{n+1} = 2y_n + x_n$. By only exchanging the words "odd" and "even", this argument also shows that $x_{n_1} = 2x_n + y_n$.

Alternatively, we can derive this second relation from the first by noting that $y_{n+1} + x_{n+1} = 3^{n+1} = 33^n = 3(y_n + x_n)$. Hence $x_{n+1} = 3(y_n + x_n) - y_{n+1} = 3(y_n + x_n) - 2y_n - x_n = y_n + 2x_n$.

(b) Simple counting shows that $y_1 = 1$ and $y_2 = 4$. From (a), we have that $y_{n+2} = 2y_{n+1} + x_{n+1}$ and $x_{n+1} = y_n + 2x_n$. It follows that $y_{n+2} = 2y_{n+1} + y_n + 2x_n$. However, $y_{n+1} = 2y_n + x_n \Rightarrow x_n = y_{n+1} - 2y_n$. Thus $y_{n+2} = 2y_{n+1} + y_n - 4y_n + 2y_{n+1} \Rightarrow y_{n+2} - 4y_{n+1} + 3y_n = 0$, as required.

(c) The characteristic equation is $r^2 - 4r + 3 = 0$ and it has solutions 3 and 1. Hence the general solution is given by $y_n = a3^n + b$, where a and b are constants determined by $y_1 = 1$ and $y_2 = 4$. We have that $3a + b = 1$ and $9a + b = 4$, which gives $a = -b = \frac{1}{2}$. The general solution is $y_n = \frac{3^n - 1}{2}$. One can also use generating functions to solve the recurrence.