General Mathematics and Computational Science I

Final Exam

December 20, 2006

1. Show that the binary operation on $\mathbb{Z}$ defined through

$$a \circ b = a + b - ab$$

   is associative, i.e. that $a \circ (b \circ c) = (a \circ b) \circ c$. (10)

2. Let $f \colon \mathbb{N} \to \mathbb{N}$ be a function with the property that $f(m+n) = f(m) + f(n)$ for all $m, n \in \mathbb{N}$. Find a formula for f and prove it by induction. (10)

3. Which of the following relations is an equivalence relation, i.e. is reflexive, symmetric, and transitive? Give explicit proofs when a relation is an equivalence relation and a counter example when it is not.

   (a) On $\mathbb{Z}$, let $a \sim b$ if and only if $a \leq b$.

   (b) On $\mathbb{Z}$, let $a \sim b$ if and only if $|a - b| \leq 10$.

   (c) Let X be a set and $U \subset X$. For any $A, B \subset X$, let $A \sim B$ if and only if $A \cap U = B \cap U$.

   (5+5+5)

4. Show that

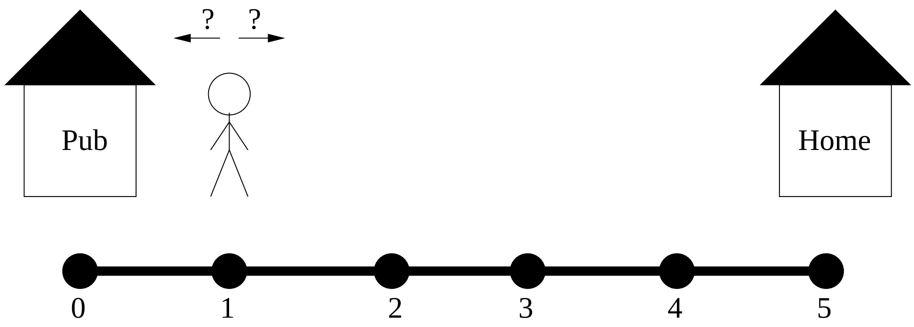
$$(a \sin\theta + b \cos\theta)^2 \leq a^2 + b^2 \,.$$

   (10)

5. A coin is tossed four times. Is it more likely to come up (a) exactly twice with the same face or (b) exactly three times with the same face? (10)

6. A drunkard lives five houses up the street from the pub. He has just left the pub in the direction of home, but has lost orientation and will move one house down or one house up with probability $\frac{1}{2}$ each. If he gets home, he will stay there. If he gets back into the pub, his friends will buy him another drink and he will stay in the pub till the next morning. What is the probability that he gets home?

   The picture shows the drunkard about to start his way home.



Hint: The probability p_n that he finds home from house no. n satisfies the difference equation $p_n = \frac{1}{2}\, p_{n-1} + \frac{1}{2}\, p_{n+1}$. Explain why. What are the boundary conditions? Then solve this difference equation. (10)

7. Consider the difference equation

$$x_{n+1} = x_n^2 - c$$

where c is a real number.

(a) Find all equilibrium points. For which values of c do equilibrium points exist?

(b) Determine the stability of the equilibrium points as a function of c.

(It is sufficient to use the derivative criterion for stability even though it is inconclusive for certain values of c.)

(10+5)

8. Prove that

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} 2^{n-k} = \frac{3^n - 1}{2}.$$

Hint: Binomial Theorem. (10)

9. Consider n-words, i.e. words of length n, from the alphabet $\{A, B, C\}$.

(a) Count the number of different n-words.

(b) Count the number of different n-words with an odd number of As.

Hint: Use the result of Question 8.

(5+10)

10. Let x_n denote the number of n-words from the alphabet $\{A, B, C\}$ with an even number of As and let y_n denote the number of such n-words with an odd number of As.

(a) Find a recurrence relation which expresses x_{n+1} and y_{n+1} in terms of x_n and y_n.

(b) Rewrite this recurrence relation as a linear second-order difference equation in y_n. What are the starting values?

Hint: You should find that

$$y_{n+2} - 4\,y_{n+1} + 3\,y_n = 0\,.$$

(c) Solve this difference equation.

(5+5+5)