Numerical Methods I – Problem Set 7

Fall Semester 2005

Due November 4

1. Let f be twice continuously differentiable and let s_L be the linear spline that interpolates f at a set of equidistant knots $x_i = x_0 + ih$ where $i = 0, \ldots, n$. Show that

$$|f(x) - s_L(x)| \leq \frac{1}{8}\, h^2 \max_{\xi \in [x_0, x_n]} |f''(\xi)|$$

for any $x \in [x_0, x_n]$.

Hint: Notice that a linear spline is determined by Lagrange interpolation on the interval between two nodes. Hence, use the error estimate for Lagrange interpolation from the lecture.

2. (From SM.) An interpolating spline of degree n has prescribed values at the knots, and is required to have $n - 1$ continuous derivatives. How many additional conditions are required to specify the spline uniquely?

3. A so-called *clamped spline* is a cubic interpolating spline with specified values of the first derivative at the first and the last knot. In the following, consider equidistant knots $x_i = i$, $i = 0, \ldots, n$. We write

$$s_i(x) = a_i\,(x - x_i)^3 + b_i\,(x - x_i)^2 + c_i\,(x - x_i) + d_i$$

to denote the spline function on the ith interval; y_i, y_i', and y_i'' denote the value of the spline, and its first and second derivative at the ith node, respectively.

(a) Show that

$$\begin{aligned} a_i &= 2\,(y_{i-1} - y_i) + y_i' + y_{i-1}'\,, \\ b_i &= 3\,(y_{i-1} - y_i) + 2\,y_i' + y_{i-1}'\,, \\ c_i &= y_i'\,, \\ d_i &= y_i\,. \end{aligned}$$

(b) Conclude from part (a) that

$$\begin{pmatrix} 4 & 1 & & & \cdots & 0 \\ 1 & 4 & 1 & & & \vdots \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & 1 & 4 & 1 \\ 0 & \cdots & & & 1 & 4 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_{n-2}' \\ y_{n-1}' \end{pmatrix} = 3 \begin{pmatrix} y_2 - y_0 \\ y_3 - y_1 \\ y_4 - y_2 \\ \vdots \\ y_{n-1} - y_{n-3} \\ y_n - y_{n-2} \end{pmatrix} - \begin{pmatrix} y_0' \\ 0 \\ 0 \\ \vdots \\ 0 \\ y_n' \end{pmatrix} .$$

4. **Project:** Solve the above system of linear equations for $n = 10$, $y_i = 0$ for all $i = 0, \ldots, 10$, $y_{10}' = 0$, and $y_0' = a$. Plot the resulting spline function. How does the spline depend on the value of a?

5. **Project:** Modify your Octave code from Homework 6 to use the Chebycheff nodes

$$x_i = 5 \cos \frac{i\,\pi}{n}$$

where $i = 0, \ldots, n$, rather than equidistant nodes, when computing the Lagrange polynomial for the function

$$f(x) = \frac{1}{1+x^2} .$$

on the interval $[-5, 5]$. What difference does it make?