Numerical Methods I – Problem Set 2

Fall Semester 2005

Due September 23, 2005

1. (Convergence acceleration). Many linearly convergent iteration schemes admit an asymptotic representation of the form

$$x_n = \xi + c\,r^n + o(r^n)$$

as $n \to \infty$, where ξ is the limit of the iteration, $|r| < 1$ is the asymptotic contraction rate, and $c \neq 0$ some constant.

   (a) Show that, for a certain value of $\alpha = \alpha(r)$, the iteration

$$y_n = \alpha\, x_n + (1 - \alpha)\, x_{n-1}$$

   converges superlinearly.

   (b) Typically, the asymptotic contraction rate r is not know *a priori*. Suggest an approximation to r based on the last three available iterates x_{n-2}, x_{n-1}, and x_n. Write out the resulting iteration scheme.

   (c) **Project:** Test your scheme on the iteration

$$x_{n+1} = \cos(x_n)\,.$$

   Use appropriate semi-logarithmic or doubly-logarithmic plots to determine the rate of convergence with and without acceleration.

2. As a rule of thumb, quadratic convergence doubles the number of accurate digits each iteration. However, this is not always true. Give an estimate on the number of accurate digits you gain each Newton iteration in terms of f' and f''.

3. Work out either (a) or (b).

   (a) Prove that the secant method converges with order $q = \frac{1}{2}(1 + \sqrt{5})$. Detailed hints are available from SM, Exercise 1.10.

   (b) Estimate, by performing an appropriate log-log plot in Octave, the rate of convergence of the secant method.

4. **Project:** Write three Octave routines to find a zero of a function $f(x)$ by using

   (a) Newton's method with starting value x_0,

   (b) the secant method with two starting values x_0 and x_1,

   (c) the bisection method on an initial interval bounded by a and b.

5. **Project:** (Example from QS.) Test your root finding routines on each of the functions

   (a) $f(x) = \cosh x + \cos x - 3$

   (b) $g(x) = \cosh x + \cos x - 2$

   and compare convergence and accuracy.