Numerical Methods I

Final Exam

December 19, 2005

For N even, the discrete Fourier transform of an N-tuple of numbers $u_0, \ldots, u_{N-1}$, is defined

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}kx_j}\, u_j$$

with $x_j = jh$ and $h = 2\pi/N$. The inverse discrete Fourier transform is given by

$$u_j = \sum_{k=-N/2}^{N/2-1} \mathrm{e}^{\mathrm{i}kx_j}\, \hat{u}_k\,.$$

1. Consider the following Octave program.

```
format long;
epsilon = 1e-14;

A = vandermonde (10); % vandermonde(n) gives the n x n Vandermonde matrix
b = sum (A,2);

[L,U,P] = lu(A);
y = L\(P*b);
x = U\y;
r = b-A*x;
[x r]

while norm(r)>epsilon
  v = L\(P*r);
  w = U\v;
  x = x + w;
  r = b-A*x;
  [x r]
end
```

   (a) What does this code do? Explain.

(b) If you run the code in Octave, you'll find the following:

```
ans =

   0.999661641799626   0.000000000000000
   1.000619472137607   0.000000016163540
   0.999521918673179  -0.000000093248673
   1.000205001753909  -0.000000229105353
   0.999945946146878  -0.000000005587935
   1.000009127129884  -0.000000104308128
   0.999999009142851  -0.000000208616257
   1.000000066933674  -0.000000119209290
   0.999999997438762   0.000000000000000
   1.000000000042420   0.000000000000000

ans =

  0.999992500612691  0.000000000000000
  1.000014060201136  0.000000000000000
  0.999988837483866  0.000000000000000
  1.000004942746758  0.000000000000000
  0.999998650834743  0.000000000000000
  1.000000235987559  0.000000000000000
  0.999999973495598  0.000000000000000
  1.000000001846354  0.000000000000000
  0.999999999927497  0.000000000000000
  1.000000000001225  0.000000000000000
```

What can you say about the condition of the problem? Is the residual a good error indicator?

(c) Will this code always terminate? Explain.

(10+10+10)

2. Use Newton's method for solving the quadratic equation $x^2 = q$ for a given positive real number q.

(a) Show that, in this case, the fixed point iteration reads

$$x_{k+1} = \frac{x_k}{2} + \frac{q}{2\,x_k} \,.$$

(b) Let x denote the exact solution. Show that

$$x_{k+1} - x = \frac{(x_k - x)^2}{2\,x_k} \,.$$

(c) What can you say about the order of convergence?

(10+5+5)

3. Compute the location of the quadrature points for Gauss quadrature on the interval $[-1,1]$ with two quadrature points.

   Hint: you may use the fact that Gauss quadrature integrates polynomials up to a certain degree exactly. (10)

4. **Note:** The remaining questions are all connected, but can be worked on independently.

   In the following, we consider the cubic interpolating spline on N equidistant grid points $x_j = jh$ for $j = 0, \dots, N-1$.

   Let y_j denote the given value of the spline on the grid nodes, and y_j'' the (unknown) value of the second derivative.

   Show that the following system of linear equations is satisfied:

$$y_{j-1}'' + 4\, y_j'' + y_{j+1}'' = \frac{6}{h^2}\,(y_{j-1} - 2\, y_j + y_{j+1})\,. \qquad (*)$$

   Hint: Use the notation from class: write

$$s_j(x) = a_j\,(x - x_j)^3 + b_j\,(x - x_j)^2 + c_j\,(x - x_j) + d_j$$

   to denote the spline function on the jth interval. Show that $2\, b_j = y_j''$ and $d_j = y_j$ and eliminate a_j and c_j by using the matching conditions at the interpolation nodes. (10)

5. Assume that the cubic spline from question 4 is defined on a periodic grid. In other words, interpolation node x_0 is identified with x_N.

   Explain why system (*) has the same number of equations as there are unkowns, i.e., why we do not need to impose additional conditions as in the case of non-periodic splines. (10)

6. Let u_j with $j = 0, \dots, N-1$ be a given tuple of numbers on an N-periodic equidistant grid with grid spacing $h = 2\pi/N$. Let $\hat{u}_k$ with $k = -N/2, \dots, N/2-1$ denote its discrete Fourier transform. Further, let τ_ℓ denote translation by ℓ grid points, i.e.

$$(\tau_\ell u)_j = u_{j+\ell}\,.$$

   Show that

$$(\widehat{\tau_\ell u})_k = \mathrm{e}^{\mathrm{i}hk\ell}\,\hat{u}_k\,.$$

(10)

7. The linear system (*) for computing the periodic spline in question 4 can be solved by taking the discrete Fourier transform on both sides of the equality.

(a) Show that in the special case that the right hand side is non-zero on a single interpolation node only,

$$y''_{j-1} + 4\,y''_j + y''_{j+1} = \delta_{0j}$$

the solution of the linear system is given by

$$y''_j = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} \frac{\mathrm{e}^{\mathrm{i}hkj}}{4 + 2\,\cos(kh)} \, .$$

Note: You are required to use the discrete Fourier transform and inverse discrete Fourier transform. The given solution is only for your convenience—direct substitution into the linear system will not earn credit.

(b) Show that the result from part (a) implies that at the gridpoint $j = N/2$,

$$|y''_{N/2}| \le \text{const} \cdot \frac{1}{N} \, .$$

Hint: First show that

$$y''_{N/2} = \frac{1}{N} \sum_{k=1}^{N/2} \frac{(-1)^k}{2 + \cos(kh)} \, .$$

(10+10)

8. Write out the linear system (*) of question 4 in matrix form *for the periodic case.*

You now know three methods for numerically solving this system:

(a) Gaussian elimination (or LU-decomposition)

(b) Iterative methods

(c) The procedure outlined in question 7, implemented in terms of the FFT and IFFT

Comment on the computational complexity and efficiency of each for this particular matrix.

(10)