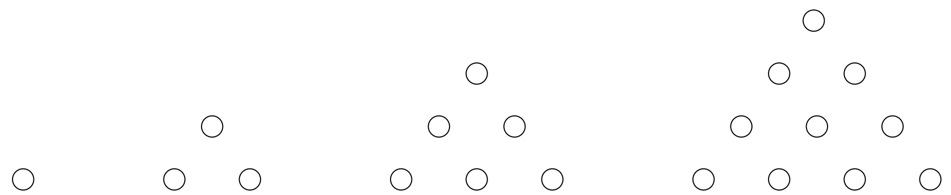
General Mathematics and Computational Science I

Midterm I

September 29, 2005

1. Let T_n denote the number of equally spaced points that fill an equilateral triangle where each side is built of n equally spaced points:



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$

Find a general formula for T_n and prove that your formula is correct. (8)

2. Show that $2^n > n^2$ for every natural number $n \geq 5$. (8)

3. Are the following functions surjective? Are they injective? Prove or disprove!

   (a) $f\colon \{1,2,3\} \to \{1,2,3\}$ where $f(1) = 2$, $f(2) = 3$, $f(3) = 3$.

   (b) Let X be a nonempty set, and $P(X)$ the set of all subsets of X, called the *power set* of X.
   Let $f\colon P(X) \to P(X)$ be defined as $f(A) = X \setminus A$.

(5+5)

4. Consider a map $G\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following properties:

(M1) $G(a, 1) = a$ for all $a \in \mathbb{N}$,

(M2) $G(a, s(b)) = G(a, b) + a$ for all $a, b \in \mathbb{N}$,

where $s\colon \mathbb{N} \to \mathbb{N}$ is as in Peano's axioms.

Prove that G is commutative, i.e.

$$G(a, b) = G(b, a)$$

for all $a, b \in \mathbb{N}$.

Hint: Consider the special case $b = 1$ first. (8)

5. Give an example of a relation on $\mathbb{N}$ which is transitive, but is neither reflexive nor symmetric.

   State explicitly why each of these properties holds respectively fails. (8)

6. Show that $\mathbb{N} \cong \mathbb{Z}$. (8)