General Mathematics and Computational Science I

Final Exam

December 16, 2005

1. Let A be a set with a finite number of elements.

   (a) Let $a \in A$. Are there more subsets of A that contain a or subsets of A that do not contain a?

   (b) If $X \subset A$, and X contains at least two elements, are there more subsets of A that contain X, or subsets of A that do not contain X?

   Explain your answer in each case. (5+5)

2. Recall from class that the rational numbers $\mathbb{Q}$ are defined as equivalence classes of tuples
$$\{[a/b] \colon a, b \in \mathbb{Z}, b \neq 0\}$$
with respect to the relation
$$[a/b] \sim [c/d] \quad \text{if } ad = bc \,. \tag{*}$$
Further, recall that $\mathbb{Q}$ can be ordered as follows. A rational number $[a/b] \in \mathbb{Q}$, is *positive* if $ab > 0$ and *negative* if $ab < 0$. Then, for rationals $x = [a/b]$ and $y = [c/d]$, we define $x > y$ if $x - y > 0$.

   (a) Verify that (*) defines an equivalence relation.

   (b) Prove, using the above definition of ordering, that for any ordered pair $x, y \in \mathbb{Q}$ with $x < y$ there exists $z \in \mathbb{Q}$ with $x < z < y$.

   (5+5)

3. Let a_n denote the sequence of Fibonacci numbers, i.e.
$$a_{n+1} = a_n + a_{n-1} \,,$$
$$a_0 = a_1 = 1 \,.$$
Prove, by induction, that
$$a_0 + a_2 + \cdots + a_{2n} = a_{2n+1} \,.$$
(10)

4. A combination lock has a five digit key; a digit can be any of the numbers $0, \ldots, 9$. What is the number of possible combinations if the digits must be in an increasing sequence? (10)

5. Use the arithmetic-geometric mean inequality to find the radius of a cylinder with prescribed surface area and the largest possible volume.

   Hints: The volume and the surface area of a cylinder of height h and radius r are given by $V = \pi h r^2$ and $S = 2\pi r^2 + 2\pi r h$, respectively. Eliminate h. (10)

6. (a) Solve the following linear programming problem using the simplex method.
   Maximize $z = 4\,x_1 + x_2$ subject to

$$x_1 + x_2 \leq 4\,,$$
$$x_2 \leq 3\,,$$
$$3\,x_1 + x_2 \leq 9\,,$$
$$x_i \geq 0 \text{ for } i = 1, 2\,.$$

   (b) Sketch the feasible region, the level lines of the objective function, and the location of the optimal point found in (a).

   (10+10)

7. Consider a Kac ring with N sites occupied by B black and W white balls. Each edge between neighboring sites carries a marker with probability μ. When the ring makes one turn, a ball just in front of a marker changes color.

   Give an expression for the probability that all balls turn white after the first turn. What happens for N large? (10)

8. If $W(B)$ denotes the number of possible ways that the sites of the Kac ring can be occupied by B black balls, define the entropy of the ring by

$$S = \ln W(B)\,.$$

   Show that when you double the number of sites, keeping the ratio of black balls constant, the entropy doubles in the limit of large N.

   Hint: Use Stirling's formula,

$$n! \sim \sqrt{2\pi n}\, n^n\, e^{-n}\,.$$

   (10)