General Mathematics and Computational Science I

Exercise 21

December 1, 2005

Ehrenfest's urn model (paraphrases from Dorfman's book). Consider two urns, I and II, and a bag. N balls, labeled $1, \dots, N$, are distributed between the urns. The bag contains N pieces of paper that carry numbers $1, \dots, N$. At each time step, somebody draws a number out of the bag at random, moves the corresponding ball from its current to the other urn, and then puts the piece of paper back into the bag. This is repeated a large number of times. Consider the difference

$$D(t) = |N_{\mathrm{I}}(t) - N_{\mathrm{II}}(t)|$$

between the number of balls in urns I and II as a function of time.

1. **Extra credit.** Write, if you can, a computer program to simulate the experiment.

2. How would you characterize the behavior of $D(t)$ for short and for long times? How can you relate this behavior to the Loschmidt and the Zermelo paradox? (A qualitative answer is sufficient.)

3. What could be the entropy for this system with its associated irreversible time evolution?

   Notice that the chance that a ball will move from the fuller urn to the emptier urn is always greater than the chance that the opposite will happen, therefore the system has the tendency to approach an equilibrium where the balls are distributed equally between the urns. Fluctuations will also occur, even very large ones.