Partial Differential Equations

Midterm Exam

October 28, 2004

1. Solve the partial differential equation

$$u_t + x\, u_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty)\,,$$
$$u = g \quad \text{on } \mathbb{R} \times \{t = 0\}\,,$$

where $u = u(x,t)$ and $g \in C^1(\mathbb{R})$.

Hint: Show that $z(s) = u(x(s), t + s)$ is constant if $x'(s) = x(s)$. (10)

2. For $U \subset \mathbb{R}^n$ open and (path-)connected, let $u \in C^2(U)$ be harmonic with $u \geq 0$. Show that if $u(x) > 0$ for some $x \in U$, then $u > 0$ everywhere in U. (10)

3. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)\,,$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x - y, t)\, g(y)\, dy\,,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|z|^2}{4t}}\,.$$

Assume that g is continuous and compactly supported. Show that there exists a $C > 0$, depending only on the support of g, such that

$$|Du(x,t)| \leq \frac{C}{\sqrt{t}}\, \|g\|_{L^\infty}\,.$$

(10)

4. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Assume that $u \in C_1^2(\bar{U} \times [0, \infty))$ solves the heat equation

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } U \times (0, \infty)\,, \\ u = g \quad & \text{on } U \times \{t = 0\}\,, \\ u = 0 \quad & \text{on } \partial U \times (0, \infty)\,, \end{aligned}$$

and define the "energy"

$$E(t) = \int_U |u(x,t)|^2 \, dx\,.$$

Prove that $E(t) \leq E(0)$ for every $t \geq 0$. (10)

5. Let $h \in C^2(\mathbb{R}^3)$ and set

$$u(x,t) = ⨍_{\partial B(x,t)} t\, h(y) \, dS(y)$$

for $t > 0$.

Show that

(a) $\lim_{t \to 0} u(x,t) = 0$.

(b) $\lim_{t \to 0} u_t(x,t) = h(x)$.

(c) **(Extra credit.)** u solves the wave equation

$$u_{tt} - \Delta u = 0$$

on $\mathbb{R}^3 \times (0, \infty)$.

(5+5+5)