1. Solve the partial differential equation

$$u_t + x u_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty),$$
$$u = g \quad \text{on } \mathbb{R} \times \{t = 0\},$$

where $u = u(x,t)$ and $g \in C^1(\mathbb{R})$.
Hint: Show that $z(s) = u(x(s), t+s)$ is constant if $x'(s) = x(s)$. (10)

$$z'(s) = x'(s)\, u_x(x(s), t+s) + u_t(x(s), t+s)$$

Therefore, if $x'(s) = x(s)$,

$$z'(s) = x(s)\, u_x(x(s), t+s) + u_t(x(s), t+s)$$
$$= (x u_x + u_t)(x(s), t+s) = 0$$

But $x'(s) = x(s) \Rightarrow x(s) = x_0 e^s$

Further: $z(0) = u(x_0, t)$
$z(-t) = u(x_0 e^{-t}, 0) = g(x_0 e^{-t})$

Since $z(0) = z(-t)$, we obtain (writing x in place of x_0),

$$u(x,t) = g(x e^{-t}) .$$

2. For $U \subset \mathbb{R}^n$ open and (path-)connected, let $u \in C^2(U)$ be harmonic with $u \geq 0$. Show that if $u(x) > 0$ for some $x \in U$, then $u > 0$ everywhere in U. (10)

Assume the contrary, i.e. $\exists\, y \in U$ s.t. $u(y) = 0$.

Let $\gamma : [0,1] \to U$ denote a path connecting x and y, i.e.
$\gamma(0) = x, \ \gamma(1) = y$

Then $\exists\, t^* \in (0,1]$ s.t. $u(\gamma(t)) = 0$ for $t \in [t^*, 1]$ (*)
$u(\gamma(t)) > 0$ for $t < t^*$ near t^*

Let $x^* = \gamma(t^*)$. By the mean value formula,

$$u(x^*) = \fint_{B(x^*, r)} u(x)\, dx$$

where r is small enough s.t. $B(x^*, r) \subset U$. Since u is continuous and $u > 0$ somewhere (along the path γ) in $B(x^*, r)$, the integral is strictly positive and $u(x^*) > 0$.
This contradicts (*).

3. Recall that the solution to the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|z|^2}{4t}}.$$

Assume that g is continuous and compactly supported. Show that there exists a $C > 0$, depending only on the support of g, such that

$$|Du(x,t)| \le \frac{C}{\sqrt{t}} \|g\|_{L^\infty}.$$

(10)

$$|Du(x,t)| = \Big| \int_{\mathbb{R}^n} \underbrace{D\Phi(x-y,t)}_{= \frac{1}{(4\pi t)^{n/2}} \left(-\frac{x-y}{2t}\right) e^{-\frac{|x-y|^2}{4t}}} g(y)\, dy \Big|$$

$$\le \frac{1}{2^{n+1}\pi^{n/2} t^{n/2+1}} \underbrace{\int_{\mathbb{R}^n} |x-y|\, e^{-\frac{|x-y|^2}{4t}}\, dy}\; \|g\|_{L^\infty}$$

$$= \int_{\mathbb{R}^n} |z|\, e^{-\frac{|z|^2}{4t}}\, dz = \int_0^\infty \int_{\partial B(0,r)} r\, e^{-\frac{r^2}{4t}}\, dz\, dr$$

$$= \int_0^\infty r\, e^{-\frac{r^2}{4t}}\, n\alpha(n)\, r^{n-1}\, dr \qquad s = \frac{r}{2\sqrt{t}} \quad \Rightarrow ds = \frac{1}{2\sqrt{t}}\, dr$$

$$= n\alpha(n) \int_0^\infty (4t)^{n/2} s^n e^{-s^2}\, 2\sqrt{t}\, ds$$

$$= \underbrace{\frac{n\alpha(n)}{\pi^{n/2}} \underbrace{\int_0^\infty s^n e^{-s^2}\, ds}_{<\infty}}_{=:C} \frac{1}{\sqrt{t}} \|g\|_{L^\infty}$$

4. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Assume that $u \in C_1^2(\bar{U} \times [0,\infty))$ solves the heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } U \times (0,\infty), \\ u &= g \quad \text{on } U \times \{t=0\}, \\ u &= 0 \quad \text{on } \partial U \times (0,\infty), \end{aligned}$$

and define the "energy"

$$E(t) = \frac{1}{2}\int_U |u(x,t)|^2\, dx.$$

Prove that $E(t) \le E(0)$ for every $t \ge 0$. (10)

Multiply the heat equation with u and integrate in space:

$$\int_U u_t u\, dx = \int_U \Delta u\, u\, dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_U |u|^2\, dx = \underbrace{\int_{\partial U} \underset{\overset{\shortparallel}{0}}{u}\, \nu\cdot Du\, dS}_{=0} - \underbrace{\int_U |Du|^2\, dx}_{\le 0}$$

$\Rightarrow E(t)$ is nonincreasing.

5. Let $h \in C^2(\mathbb{R}^3)$ and set

$$u(x,t) = \fint_{\partial B(x,t)} t\, h(y)\, dS(y)$$

for $t > 0$.
Show that

(a) $\lim_{t \to 0} u(x,t) = 0$.

(b) $\lim_{t \to 0} u_t(x,t) = h(x)$.

(c) **(Extra credit.)** u solves the wave equation

$$u_{tt} - \Delta u = 0$$

on $\mathbb{R}^3 \times (0, \infty)$.

(5+5+5)

$y = x + tz$

$\Rightarrow dS(y) = t^2\, dS(z)$

$$= t \frac{1}{3\alpha(3)\, t^2} \int_{\partial B(x,t)} h(y)\, dS(y)$$

$$= \frac{t}{3\alpha(3)} \int_{\partial B(0,1)} h(x+tz)\, dS(z) \qquad (*)$$

(a) $$u(x,t) = t \underbrace{\fint_{\partial B(x,t)} h(y)\, dS(y)}_{\to h(x) \text{ as } t \to 0} \to 0 \quad \text{as } t \to 0$$

(b) $$u_t = \fint_{\partial B(x,t)} h(y)\, dS(y) + \frac{t}{3\alpha(3)} \underbrace{\int_{\partial B(0,1)} z \cdot Dh(x+tz)\, dS(z)}$$

$$= \frac{1}{t^2} \int_{\partial B(x,t)} \underbrace{\frac{y-x}{t}}_{=\nu} \cdot Dh(y)\, dS(y)$$

$$= \frac{1}{t^2} \int_{B(x,t)} \Delta h\, dy$$

$$= \underbrace{\fint_{\partial B(x,t)} h(y)\, dS(y)}_{\to h(x)} + \underbrace{\frac{t^2}{3} \underbrace{\fint_{B(x,t)} \Delta h\, dy}_{\to \Delta h(x)}}_{\to 0}$$

(c) $$u_t = \underbrace{\frac{1}{3\alpha(3)\, t^2} \int_{\partial B(x,t)} h(y)\, dS(y)}_{= \frac{1}{3\alpha(3)} \int_{\partial B(0,1)} h(x+tz)\, dS(z)} + \frac{1}{3\alpha(3)\, t} \int_0^t \int_{\partial B(x,s)} \Delta h\, dS(y)\, ds$$

$$\Rightarrow u_{tt} = \frac{1}{3\alpha(3)} \int_{\partial B(0,1)} z \cdot Dh(x+tz)\, dS(z) - \frac{1}{3\alpha(3)} \frac{1}{t^2} \int_{B(x,t)} \Delta h\, dy$$

$$+ \frac{1}{3\alpha(3)\, t} \int_{\partial B(x,t)} \Delta h\, dS(y)$$

Changing back from z to y in the first integral, and applying the Gauss integral formula as in (b), we see that the first two terms cancel.

$$\Rightarrow u_{tt} = \frac{1}{3\alpha(3)\, t} \int_{\partial B(x,t)} \Delta\, dS(y)$$

On the other hand, taking Δ of $(*)$,

$$\Delta u = \frac{t}{3\alpha(3)} \int_{\partial B(0,1)} \Delta h(x+tz)\, dS(z)$$

$$= \frac{t}{3\alpha(3)\, t^2} \int_{\partial B(x,t)} \Delta h\, dS(y)$$

$$\Rightarrow u_{tt} = \Delta u$$

□