1. Consider a modified Laplace equation,

$$D \cdot (r\,Du(x)) = 0\,,$$

where $u\colon \mathbb{R}^n \to \mathbb{R}$ for $n \geq 2$, and $r = |x|$.

(a) Show that if u is a radial solution, i.e. $u(x) = v(r)$, then

$$r v'' + n v' = 0\,.$$

(b) Conclude that any radial solution with $u(r) \to 0$ as $r \to \infty$ is a multiple of

$$u(x) = \frac{1}{r^{n-1}}\,.$$

(5+5)

(a) Recall that $Dr = \frac{x}{r}$

$\Rightarrow \quad Dv(r) = v' \frac{x}{r}$

$\Rightarrow \quad r\,Dv(r) = v' x$

$\Rightarrow \quad D\cdot(r\,Dv(r)) = x \cdot \underbrace{Dv'}_{= v'' \frac{x}{r}} + v' \underbrace{D\cdot x}_{=n} = r v'' + n v' \equiv 0$

(b) From (a):

$\frac{dv'}{v'} = -n \frac{dr}{r} \quad \Rightarrow \quad \ln v' = -n \ln r + c = \ln r^{-n} + c$

$\Rightarrow \quad v' = e^{c}\, r^{-n}$

Integrate once more:

$v = \frac{1}{1-n} e^{c}\, r^{1-n} + \tilde{c}\,,$

Since $v(r) \to 0$ as $r \to \infty$, $\tilde{c} = 0$.

Since the equation is homogeneous and linear, any multiple of this solution is also a solution. $\square$

2. Let $u(x,t) \geq 0$ denote a smooth temperature distribution within a homogeneous, heat-conducting medium U. Hence, u satisfies the heat equation. It is known that the heat energy contained in U is proportional to

$$E = \int_U u(x,t)\,dx\,.$$

(a) Show that the boundary condition $u(x,t) = 0$ for $x \in \partial U$ corresponds to cooling, i.e. that the heat energy is non-increasing.

(b) What boundary condition would you use to describe a perfectly insulated medium?

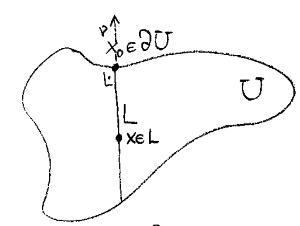
(c) **Extra credit:** Prove that in case (a), provided U is bounded, the heat energy decreases exponentially.

(5+5+*10*)

(a) $\frac{dE}{dt} = \int_U u_t\,dx = \int_U \Delta u\,dx = \int_{\partial U} \nu \cdot Du\,dS$

Since $u \geq 0$ in U and $u = 0$ on $\partial\Omega$ (the maximum principle ensures that this property is maintained if true initially),

$$\nu \cdot Du(x_0) = \lim_{\substack{x \to x_0 \\ x \in L}} \frac{u(x_0) - u(x)}{|x_0 - x|} \leq 0$$



$\Rightarrow \quad \frac{dE}{dt} \leq 0\,.$

(b) Insulation means that $\frac{dE}{dt} = 0$, so we choose the "no heat flux" boundary condition

$$\nu \cdot Du = 0 \quad \text{on } \partial\Omega .$$

(c) Consider $\quad H = \int_U u^2 \, dx$

$$\Rightarrow \frac{1}{2}\frac{dH}{dt} = \int_U u \Delta u \, dx = - \int_U |Du|^2 \, dx$$

$$\overset{\circledast}{\leq} c H$$

$$\Rightarrow \quad H(t) \leq H(0)\, e^{-ct}$$

Moreover,

$$0 \leq \int_U u \, dx \leq \underbrace{\left(\int_U dx \right)^{\frac{1}{2}}}_{\substack{\leq \tilde{c} \text{ since} \\ U \text{ is bounded}}} \underbrace{\left(\int_U u^2 \, dx \right)^{\frac{1}{2}}}_{\leq H_0^{\frac{1}{2}} e^{-\frac{1}{2}ct}}$$

$\Rightarrow$ $E(t)$ decreases exponentially.

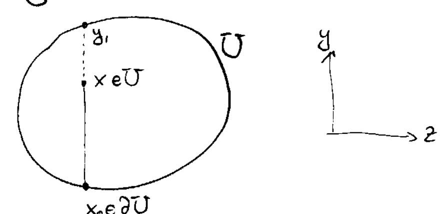
$\circledast$ This step depends on the Poincaré inequality for functions that vanish on the boundary. It is known to be true whenever U is bounded.

A conceptionally nice proof uses results on the spectrum of the Dirichlet-Laplacian on bounded domains which are beyond the scope of this course (on a bounded domain, the spectrum of Δ with $u = 0$ on ∂U consists only of eigenvalues that are negative, countable, and bounded away from 0; the set of eigenfunctions is complete).

Here a more elementary proof:

First, assume that U is convex, and write $x = (\underset{\in \mathbb{R}}{y}, \underset{\in \mathbb{R}^{n-1}}{z})$.

Then for every $x = (y,z) \in U$ there exists an $x_0 = (y_0, z) \in \partial U$ such that the line connecting x and x_0 is contained in U:



$$\Rightarrow \quad u^2(x) - \underbrace{u^2(x_0)}_{=0} = \int_{y_0}^{y} \partial_y (u^2) \, d\tilde{y} \leq 2 \int_{y_0}^{y_1} |u| \, |u_y| \, dy$$

$$\leq 2 \left(\int_{y_0}^{y_1} u^2 \, dy \right)^{\frac{1}{2}} \left(\int_{y_0}^{y_1} u_y^2 \, dy \right)^{\frac{1}{2}} \quad (*)$$

Now, integrate over $\bar{U} = \{ (y,z) : y_0(z) < y < y_1(z) ; \; z \in U_z \}$

$$\int_U u^2(x)\,dx = \int_{U_z} \int_{y_0(z)}^{y_1(z)} u^2(y,z)\,dy\,dz$$

$$\overset{(*)}{\leq} 2 \int_{U_z} \underbrace{\int_{y_0(z)}^{y_1(z)} dy}_{\leq \operatorname{diam} U} \left(\int_{y_0(z)}^{y_1(z)} u^2\,dy \right)^{\frac{1}{2}} \left(\int_{y_0(z)}^{y(z)} u_y^2\,dy \right)^{\frac{1}{2}} dz$$

$$\overset{C.S.}{\leq} 2 \operatorname{diam} U \left(\int_{U_z} \int_{y_0(z)}^{y_1(z)} u^2\,dy\,dz \right)^{\frac{1}{2}} \left(\int_{U_z} \int_{y_0(z)}^{y_1(z)} u_y^2\,dy\,dz \right)^{\frac{1}{2}}$$

$$= 2 \operatorname{diam} U \underbrace{\left(\int_U u^2\,dx \right)^{\frac{1}{2}}}_{= \|u\|_{L^2}} \underbrace{\left(\int_U u_y^2\,dx \right)^{\frac{1}{2}}}_{\leq \|Du\|_{L^2}}$$

$$\Rightarrow \quad \|u\|_{L^2} \leq 2 \operatorname{diam} U \ \|Du\|_{L^2} \ .$$

This result extends directly with a different constant, to any domain which is diffeomorphic to a convex domain. $\square$

3. (a) Let $U \in \mathbb{R}^n$ be open and simply connected, with sufficiently smooth boundary, and assume that the *Poincaré inequality*

$$\|u\|_{L^2} \leq C\,\|Du\|_{L^2} \qquad (*)$$

holds for some constant C and for every $u \in H^1(U)$ with $u = 0$ on ∂U. Show that

$$C \geq \frac{1}{\sqrt{|\lambda|}},$$

where λ is an eigenvalue of the Laplacian corresponding to an eigenfunction that vanishes on ∂U.

(b) Show that there is no Poincaré inequality for $U = \mathbb{R}^n$.

(5+5)

(a) Let u be such an eigenfunction, i.e.

$$\Delta u = \lambda u \quad , \qquad u = 0 \ \text{ on } \partial U$$

$$\Rightarrow \underbrace{\int_U u\,\Delta u\,dx}_{= -\int_U |Du|^2\,dx} = \lambda \int_U u^2\,dx$$

Since both integrals are positive, $\lambda < 0$, and

$$\frac{1}{\sqrt{|\lambda|}} \|Du\|_{L^2} = \|u\|_{L^2}$$

Thus, if $C < \frac{1}{\sqrt{|\lambda|}}$, this would violate $(*)$. $\square$

(b) Consider the radial function

$$u_\varepsilon(r) = \max\left\{0,\ 1 - \varepsilon r\right\}$$

$$\Rightarrow \quad Du_\varepsilon = \begin{cases} -\varepsilon \frac{x}{r} & \text{for } 0 < r < \frac{1}{\varepsilon} \\ 0 & \text{for } r > \frac{1}{\varepsilon} \end{cases}$$

$$\Rightarrow \quad \int_{\mathbb{R}^n} u_\varepsilon^2 \, dx = \int_0^{\frac{1}{\varepsilon}} (1 - \varepsilon r)^2 \underbrace{\int_{\partial B(0,r)} dS}_{= n\alpha(n) r^{n-1}} \, dr$$

$$= n\alpha(n) \int_0^{\frac{1}{\varepsilon}} \left(r^{n-1} - 2\varepsilon r^n + \varepsilon^2 r^{n+1}\right) dr$$

$$= n\alpha(n) \left(\frac{1}{n}\varepsilon^{-n} - \frac{2\varepsilon}{n+1}\varepsilon^{-n+1} + \frac{\varepsilon^2}{n+2}\varepsilon^{-n+2}\right)$$

$$= c(n)\, \varepsilon^{-n}$$

$$\int_{\mathbb{R}^n} |Du_\varepsilon|^2 \, dx = \varepsilon^2 \int_{B(0,\frac{1}{\varepsilon})} dx = \varepsilon^2 \alpha(n) \varepsilon^{-n}$$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} \frac{\int_{\mathbb{R}^n} |u_\varepsilon|^2 dx}{\int_{\mathbb{R}^n} |Du_\varepsilon|^2 dx} \longrightarrow \infty \quad \text{as } \varepsilon \to 0$$

Hence, there cannot be a constant C that bounds this ratio.

4. Assume that $U \subset \mathbb{R}^n$ is open and connected. Show that if $u \in C^2(U)$ solves the *Neumann problem*

$$\begin{aligned} -\Delta u = 0 \quad & \text{in } U, \\ \nu \cdot Du = 0 \quad & \text{on } \partial U, \end{aligned}$$

then $u = \text{const.}$ (10)

$$-\Delta u = 0 \quad \Rightarrow \quad -\int_U u \,\Delta u \, dx = 0$$

$$\Rightarrow \quad \int_U |Du|^2 \, dx - \int_{\partial U} u \underbrace{\nu \cdot Du}_{=0} \, dS = 0$$

$$\Rightarrow \quad Du = 0 \text{ in } U$$

$$\Rightarrow \quad u = \text{const}$$

5. *Note:* The different parts of this question can be worked on independently. If you get stuck, move on.

(a) Let $u \in H^1([0,1])$ with

$$u(0) = u(1) = 0 .$$

Prove that

$$|u^m(x)| \le m \left(\int_0^1 |u|^m \, dx\right)^{\frac{1}{2}} \left(\int_0^1 |u|^{m-2} |u_x|^2 \, dx\right)^{\frac{1}{2}}$$

Hint: Apply the Fundamental Theorem of Calculus to $v = u^m$; Cauchy–Schwarz inequality.

(b) Deduce from (a) that

$$\int_0^1 |u|^{2m} \, dx \le \frac{m}{2} \left(\int_0^1 |u|^m \, dx\right)^3 + \frac{m}{2} \int_0^1 |u|^{m-2} |u_x|^2 \, dx .$$

(c) Consider the partial differential equation

$$u_t = u_{xx} + u^{m+1}$$
$$u(0) = u(1) = 0$$

on $U = (0,1) \times (0,T)$, where $m \ge 2$ is an even integer. Show that

$$\frac{1}{m} \frac{d}{dt} \int_0^1 u^m \, dx = -(m-1) \int_0^1 u^{m-2} u_x^2 \, dx + \int_0^1 u^{2m} \, dx .$$

(d) Combine (b) and (c) to find that

$$\frac{d}{dt} \int_0^1 u^m \, dx \le \frac{m^2}{2} \left(\int_0^1 |u|^m \, dx\right)^3 .$$

Finally, conclude that if $u(0) \in L^m([0,1])$, then $u(t) \in L^m([0,1])$ for every $t \in [0,T)$. Give a lower bound for T.

(5+5+5+5)

(a)

$$\underbrace{U^m(x) - U^m(0)}_{=0} = \int_0^x \partial_\xi U^m(\xi)\, d\xi = m \int_0^x U^{m-1} U_\xi \, d\xi$$

$$\Rightarrow |U^m(x)| \le m \int_0^1 |U|^{m-1} |U_x| \, dx$$

$$= m \int_0^1 |U|^{\frac{m}{2}} |U|^{\frac{m}{2}-1} |U_x| \, dx$$

$$\overset{C.S.}{\le} m \left(\int_0^1 |U|^m dx\right)^{\frac{1}{2}} \left(\int_0^1 |U|^{m-2} |U_x|^2 \, dx\right)^{\frac{1}{2}} \qquad \square$$

(b)

$$\int_0^1 |U|^{2m} dx \le \int_0^1 |U|^m dx \; \|U\|^m_{L^\infty}$$

$$\overset{(a)}{\le} m \left(\int_0^1 |U|^m dx\right)^{\frac{1}{2}} \left(\int_0^1 |U|^{m-2} |U_x|^2 dx\right)^{\frac{1}{2}}$$

$$= m \left(\int_0^1 |U|^m dx\right)^{\frac{3}{2}} \left(\int_0^1 |U|^{m-2} |U_x|^2 dx\right)^{\frac{1}{2}}$$

$$\overset{\text{Young's}}{\le} m \left[\frac{1}{2}\left(\int_0^1 |U|^m dx\right)^3 + \frac{1}{2} \int_0^1 |U|^{m-2} |U_x|^2 dx \right] \qquad \square$$

(c) Multiply equation by U^{m-1} and integrate:

$$\underbrace{\int_0^1 U^{m-1} U_t \, dx}_{= \frac{1}{m} \frac{d}{dt} \int_0^1 U^m dx} = \underbrace{\int_0^1 U^{m-1} U_{xx} \, dx}_{= \underbrace{U^{m-1} U_x \Big|_0^1}_{=0} - \int_0^1 (m-1) U^{m-2} U_x U_x \, dx} + \underbrace{\int_0^1 U^{m-1} U^{m+1} \, dx}_{= \int_0^1 U^{2m} dx} \qquad \square$$

(d)

$$\frac{1}{m} \frac{d}{dt} \int_0^1 U^m dx \le \underbrace{-(m-1) \int_0^1 U^{m-2} U_x^2 \, dx + \frac{m}{2} \int_0^1 U^{m-2} U_x^2 \, dx}_{\le 0 \text{ for } m \ge 2} + \frac{m}{2}\left(\int_0^1 U^m dx\right)^3$$

$$\Rightarrow \frac{d}{dt} \underbrace{\int_0^1 U^m dx}_{=:y(t)} \le \frac{m^2}{2} \left(\int_0^1 U^m dx\right)^3$$

$$\Rightarrow \dot{y} \le \frac{m^2}{2} y^3$$

$$\Rightarrow \quad \frac{dy}{y^3} \leq \frac{m^2}{2}\, dt$$

$$\Rightarrow \quad \int_{y_0}^{y(t)} \frac{dy}{y^3} \leq \frac{m^2}{2} \int_0^t dt$$

$$\Rightarrow \quad -\frac{1}{2} y^{-2} \Big|_{y_0}^{y(t)} \leq \frac{m^2}{2}\, t$$

$$\Rightarrow \quad y_0^{-2} - y^{-2}(t) \leq m^2 t$$

$$\Rightarrow \quad y^2(t) \leq \frac{1}{y_0^{-2} - m^2 t}$$

We hence obtain a bound which blows up at

$$y_0^{-2} = m^2 T$$

$$\Rightarrow T = \frac{1}{m^2 y_0^2}$$

$$= \frac{1}{m^2} \frac{1}{\left(\int_0^1 u_0^m(x)\, dx \right)^2} .$$