Partial Differential Equations

Homework 8

due November 30, 2004

In the following, $\mathbb{T}$ denotes the 1-torus, i.e. $\mathbb{T} = \mathbb{R} \bmod 2\pi$.

1. (a) Show that, for every $u \in H^2(\mathbb{T})$,

$$\|u\|_{H^1}^2 \leq \|u\|_{L^2} \, \|u\|_{H^2} \, .$$

   (b) Consider the Fisher–Kolmogorov equation on $\mathbb{T}$,

$$u_t = u_{xx} + (1-u)\, u^m \, ,$$
$$u(0) = u^{\text{in}} \, ,$$

   where m is an even positive integer. Use the result from (a), as well as the first question of the previous homework set, to prove that that

$$\limsup_{t \to \infty} \|u(t)\|_{H^1} \leq C$$

   where an explicit estimate for C can be given which, in particular, shows that C does not depend on the initial data u^{in}. You may assume that u is sufficiently differentiable so that all your formal manipulations are justified.

2. Prove the following version of the *Poincaré inequality*: For every $u \in H^1(\mathbb{T})$ which has zero mean, i.e. where

$$⨍_{\mathbb{T}} u \, dx = 0 \, ,$$

   we have

$$\int_{\mathbb{T}} |u|^2 \, dx \leq C \int_{\mathbb{T}} |u_x|^2 \, dx \, .$$

   Find the best estimate for C.

3. (a) Consider a sequence $u_n \in L^2$ with $u_n \rightharpoonup u \in L^2$ weakly. Show that

$$\|u\| \leq \liminf_{n \to \infty} \|u_n\| \, . \qquad (*)$$

   (Remark: This statement is actually true for any Banach space.)

(b) Give an example where (*) holds with strict inequality.

4. Consider the inviscid Burger's equation on $\mathbb{T}$, i.e.

$$u_t + u\, u_x = 0\,.$$

(a) Define an approximate solution u_n by applying the projector $\mathbb{P}_n$ which projects onto modes up to wave number n to Burger's equation. Show that

$$\|u_n(t)\|_{L^2} = \|u_n(0)\|_{L^2}\,.$$

(b) Conclude that $\{u_n\}$ has a subsequence that converges to some u weakly in $L^2(\mathbb{T})$, and that

$$\|u(t)\|_{L^2} \leq \|u(0)\|_{L^2}\,.$$

Why do you have an inequality rather than equality?

(Note: you are not required to show that u solves Burger's equation in any sense. This would require a much more involved analysis.)

Grading: 6 points per question; there is a penalty of 1 point per day on late submissions!