Partial Differential Equations

Homework 5

due October 14, 2004

1. Finish the proof of the mean value formula for the heat equation by showing that

$$\iint_{E(0,0;1)} \frac{|y|^2}{s^2}\, dy\, ds = 4\,,$$

where

$$E(x,t;r) = \{(y,s) \colon \Phi(x-y,t-s) \geq r^{-n}\}$$

denotes the heat ball "centered" at (x,t).

Hint: Use polar coordinates in space, and an appropriate change of variables in time. The remaining one-dimensional integral is MATHEMATICA-integrable. You can also use that

$$\int_0^\infty t^{\lambda+1}\, e^{-\lambda t}\, dt = \frac{\Gamma(\lambda+2)}{\lambda^{2+\lambda}}\,,$$

$$\Gamma(x+1) = x\,\Gamma(x)\,,$$

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,.$$

2. Prove a maximum principle for the following semilinear PDE, called Burger's equation,

$$u_t + u\, u_x = u_{xx}\,,$$
$$u(x,0) = g(x)$$

where $u = u(x,t)$ and $(x,t) \in \mathbb{R} \times [0,\infty)$.

3. Evans, p. 87 problem 12

4. Evans, p. 87 problem 13

Grading: 5 points per question; there is a penalty of 1 point per day on late submissions!