Homework 9 solutions

1. (a)

$$0 = T_n(x) = \cos(n \arccos x)$$

$$\Rightarrow \quad n \arccos x = \frac{\pi}{2} + j\pi \qquad , j = 0, \dots, n-1$$

$$\Rightarrow \quad x = \cos \frac{(1+2j)\pi}{2n}$$

are the zeros of T_n.

To find the extrema, we check the necessary condition

$$0 = T_n'(x) = -\sin(n \arccos x) \cdot n \arccos'(x)$$

$$= n \frac{\sin(n \arccos x)}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad n \arccos x = j\pi \qquad , j = 1, \dots, n-1$$

$$\Rightarrow \quad x = \cos \frac{j\pi}{n}$$

The endpoints of the interval are obviously extrema as well, so the set of extrema is

$$x_j = \cos \frac{j\pi}{n} \qquad , \quad j = 0, \dots, n$$

(b)

$$\langle T_n, T_m \rangle = \int_{-1}^{1} \frac{\cos(n \arccos x) \cos(m \arccos x)}{\sqrt{1-x^2}} \, dx$$

substitute $\quad x = \cos \phi$

$$\Rightarrow dx = -\sin \phi \, d\phi$$

$$1 - x^2 = \sin^2 \phi$$

$$\Rightarrow \langle T_n, T_m \rangle = \int_{\pi}^{0} \frac{\cos(n\phi) \cos(m\phi)}{\sin \phi} (-\sin \phi) \, d\phi$$

$$= \int_{0}^{\pi} \cos(n\phi) \cos(m\phi) \, d\phi$$

$$= \frac{1}{2} \underbrace{\int_{0}^{\pi} \cos((n+m)\phi) \, d\phi}_{\substack{=0 \text{ if } n \neq 0 \text{ or } m \neq 0 \\ \text{by symmetry}}} + \frac{1}{2} \underbrace{\int_{0}^{\pi} \cos((n-m)\phi) \, d\phi}_{= \pi \delta_{nm}}$$

$$= \frac{\pi}{2} \delta_{nm}$$

□

2. $$\frac{dy}{dt} = y - y^2$$

$$\Rightarrow \frac{dy}{y(1-y)} = dt$$

$$\Rightarrow \int_{y_0}^{y(t)} \underbrace{\frac{dy}{y(1-y)}}_{} = \int_0^t dt$$

$$= \frac{dy}{y} + \frac{dy}{1-y} \qquad \text{(partial fractions)}$$

$$\Rightarrow \ln y - \ln(1-y) \Big|_{y_0}^{y(t)} = t$$

$$\Rightarrow \ln\left(\frac{y}{1-y} \cdot \frac{1-y_0}{y_0}\right) = t$$

$$\Rightarrow \frac{y}{1-y} = \frac{y_0}{1-y_0} e^t$$

$$\frac{y}{1-y} = a \Rightarrow y = a - ya \Rightarrow y(1+a) = a \Rightarrow y = \frac{a}{1+a}$$

$$\Rightarrow y(t) = \frac{\frac{y_0}{1-y_0} e^t}{1 + \frac{y_0}{1-y_0} e^t}$$

$$= \frac{y_0}{y_0 + (1-y_0) e^{-t}}$$

3. (a) The eigenvalues of A are obviously $\lambda_1 = 1$, $\lambda_2 = 3$.

Eigenvector for λ_1:

$$\begin{pmatrix} (1-1) & 2 \\ 0 & (3-1) \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Eigenvector for λ_2

$$\begin{pmatrix} (1-3) & 2 \\ 0 & (3-3) \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$A = S D S^{-1}$$

$$\Rightarrow e^A = S e^D S^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} e & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{= \begin{pmatrix} e & -e \\ 0 & e^3 \end{pmatrix}} = \begin{pmatrix} e & e^3 - e \\ 0 & e^3 \end{pmatrix}$$

(b) Find the eigenvalues:

$$\det(A-\lambda I) = \det\begin{pmatrix} a-\lambda & b \\ -b & a-\lambda \end{pmatrix} = (a-\lambda)^2 + b^2 \equiv 0$$

$$\Rightarrow (a-\lambda)^2 = -b^2$$

$$\Rightarrow a-\lambda = \pm ib$$

$$\Rightarrow \lambda = a \mp ib$$

Eigenvector for $\lambda_1 = a+ib$:

$$\begin{pmatrix} a-(a+ib) & b \\ -b & a-(a+ib) \end{pmatrix} \to \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$$

Eigenvector for $\lambda_2 = a-ib$:

$$\begin{pmatrix} a-(a-ib) & b \\ -b & a-(a-ib) \end{pmatrix} \to \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \to \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\Rightarrow D = \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix}$$

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$$

Note: We normalize so that S is unitary, and therefore

$$S^{-1} = S^H = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$$

Then

$$A = SDS^{-1}$$

and

$$e^A = S e^D S^{-1}$$

$$= \frac{1}{2}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} e^{a+ib} & 0 \\ 0 & e^{a-ib} \end{pmatrix}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}}_{= \begin{pmatrix} -ie^{a+ib} & -e^{a+ib} \\ -ie^{a-ib} & e^{a-ib} \end{pmatrix}}$$

$$= \frac{1}{2} e^a \begin{pmatrix} e^{ib} + e^{-ib} & -ie^{ib} + e^{-ib} \\ ie^{ib} - ie^{-ib} & e^{ib} + e^{-ib} \end{pmatrix}$$

$$= e^a \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}$$

4. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$$e^{A+B} = \begin{pmatrix} e & e^3 - e \\ 0 & e^3 \end{pmatrix} \quad \text{from } 3(a).$$

<u>But:</u> $e^A = \begin{pmatrix} e & 0 \\ 0 & e^3 \end{pmatrix}$, $e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!} = I + B + \frac{1}{2}\underbrace{B^2}_{=0} + 0 + \ldots = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow e^A e^B = \begin{pmatrix} e & 2e + e^3 \\ 0 & e^3 \end{pmatrix} \neq e^{A+B}.$$