Homework 8 solutions

1. (a) We test the formula using the basis polynomials $1, x, x^2, \ldots$

$$f=1: \int_{-1}^{1} dx = 2 \qquad w_0 f(-\alpha) + w_1 f(\alpha) = w_0 + w_1$$

$$f=x: \int_{-1}^{1} x\,dx = 0 \qquad w_0 f(-\alpha) + w_1 f(\alpha) = -\alpha w_0 + \alpha w_1$$

$$f=x^2: \int_{-1}^{1} x^2\,dx = \frac{2}{3} \qquad w_0 f(-\alpha) + w_1 f(\alpha) = \alpha^2 w_0 + \alpha^2 w_1$$

$$f=x^3: \int_{-1}^{1} x^3\,dx = 0 \qquad w_0 f(-\alpha) + w_1 f(\alpha) = \alpha^3 (w_1 - w_0)$$

If the formula is exact for polynomials up to degree 1, then

$$\left.\begin{array}{l} w_0 + w_1 = 2 \\ \alpha(w_1 - w_0) = 0 \quad \Rightarrow \quad w_1 = w_0 \end{array}\right\} w_1 = w_0 = 1$$

(b) If the formula is also exact for polynomials of degree 2,

$$\alpha^2 (w_0 + w_1) = \frac{2}{3} \quad \Rightarrow \quad \alpha = \sqrt{\frac{1}{3}}$$

Clearly, $\alpha^3 (w_1 - w_0) = 0$, so the formula is exact for polynomials of degree 3 as well.

2. (a) On the first partition $[0,h]$:

$$E_1 = \int_0^h x^{\frac{1}{3}}\,dx - h\,\frac{0^{\frac{1}{3}} - h^{\frac{1}{3}}}{2}$$

$$= \frac{3}{4} h^{\frac{4}{3}} - \frac{1}{2} h^{\frac{4}{3}} = \frac{1}{4} h^{\frac{4}{3}}$$

(b) The trapezoidal rule on each interval $[(i-1)h, ih]$, $i \geq 2$, is identical to exact integration of the interpolating polynomial of degree 1. We thus use the standard error estimate for Lagrange interpolation:

$$|E_i| = \left| \int_{(i-1)h}^{ih} f(x)\,dx - h\,\frac{f((i-1)h) + f(ih)}{2} \right|$$

$$\leq \left| \int_{(i-1)h}^{ih} \frac{M_2}{2!} (x-(i-1)h)(x-ih)\,dx \right|$$

$$= \underbrace{\frac{1}{2} \max_{x \in [(i-1)h, ih]} \underbrace{|f''(x)|}_{= \frac{1}{3}\frac{2}{3} x^{-\frac{5}{3}}}}_{= \frac{1}{9}[(i-1)h]^{-\frac{5}{3}}} \; \underbrace{\left| \int_{(i-1)h}^{ih} (x-(i-1)h)(x-ih)\,dx \right|}_{\substack{= \int_0^h x(x-h)\,dx \\ = \frac{1}{3}x^3 - \frac{1}{2}hx^2 \Big|_0^h \\ = -\frac{1}{6} h^3}}$$

$$\Rightarrow \quad |E_i| \leq \frac{1}{54}\,(i-1)^{-\frac{5}{3}}\, h^{\frac{4}{3}}$$

so that the error of all partitions $2, \dots, n$ is bounded by

$$E_{rest} \equiv \sum_{i=2}^{n} \frac{1}{54}\,(i-1)^{-\frac{5}{3}}\, h^{\frac{4}{3}} = \frac{h^{\frac{4}{3}}}{54} \underbrace{\sum_{i=1}^{n-1} i^{-\frac{5}{3}}}_{\leq\, 1 + \underbrace{\int_1^\infty x^{-\frac{5}{3}}\,dx}_{= -\frac{3}{2} x^{-\frac{2}{3}} \Big|_1^\infty = \frac{3}{2}} \,=\, \frac{5}{2}} \leq \frac{5}{108}\, h^{\frac{4}{3}}$$

Since $\frac{5}{108} < \frac{1}{4}$, it is not possible to cancel the error E_1 made in the first partition with any of the errors made in the other partition, i.e. the total error is of order $h^{\frac{4}{3}}$. The Euler-Maclaurin summation formula does not apply because $x^{\frac{1}{3}}$ is not differentiable at $x = 0$.

4. The Euler-Maclaurin summation formula is (in this case)

$$\int_{-1}^{1} f(x)\,dx - T(f) = \sum_{\substack{j=2 \\ j\,\text{even}}}^{k} q_j(1)\left(\frac{h}{2}\right)^j \left(f^{(j-1)}(1) - f^{(j-1)}(-1)\right) - \left(\frac{h}{2}\right)^k \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} q_k(x)\, f^{(k)}(x)\,dx$$

- when $f(x) = \sinh x$,

  $f^{(j-1)}(x) = \cosh x$ for j even

  $\Rightarrow f^{(j-1)}(1) - f^{(j-1)}(-1) = 0$

  we expect arbitrary order of accuracy.

- when $f(x) = \cosh x$,

  $f^{(j-1)}(x) = \sinh x$ for j even

  $\Rightarrow f^{(j-1)}(1) - f^{(j-1)}(-1) = 2 \sinh 1 \neq 0$

  we expect only order 2 convergence.

6. Gauss integration is exact whenever f is a polynomial of degree up to $2n+1$, and has the general formula

$$\int_a^b w(x)\, f(x)\, dx \approx \sum_{k=0}^{n} W_k\, f(x_k) \qquad (*)$$

(note that all the weights V_k are chosen to be zero.)

On the other hand, the Newton-Cotes-like formula on the Gauss quadrature points $x_0, \dots, x_n$ is obtained by replacing f by its Lagrange interpolating polynomial,

$$\int_a^b w(x)\, f(x)\, dx \approx \int_a^b w(x) \sum_{k=0}^{n} L_k(x)\, f(x_k)\, dx$$

$$\equiv \sum_{k=0}^{n} \tilde{W}_k\, f(x_k) \qquad (**)$$

where $\tilde{W}_k = \int_a^b w(x)\, L_k(x)\, dx$.

This formula is exact for all polynomials of degree (at least) up to n; we therefore expect that this provides $n+1$ conditions that uniquely determine $\tilde{W}_0, \dots, \tilde{W}_n$, and that therefore $W_k = \tilde{W}_k$, $k = 0, \dots, n$.

To prove the uniqueness of the choice of weights, choose $f(x) = L_j(x)$ which is integrated exactly by both $(*)$ and $(**)$. Thus,

$$\sum_{k=0}^{n} W_k \underbrace{L_j(x_k)}_{=\delta_{jk}} = \sum_{k=0}^{n} \tilde{W}_k \underbrace{L_j(x_k)}_{=\delta_{jk}}$$

$$\Rightarrow W_j = \tilde{W}_j$$

$\square$