Homework 7 solutions

1. On each of the partitions $[x_{j-1}, x_j]$ for $j=1,\dots,n$ the linear spline is identical to the Lagrange interpolating polynomial of degree 1. Thus, on the j-th partition,

$$|f(x) - S_L(x)| \leq \frac{M_2}{2!} \, |(x-x_{j-1})(x-x_j)|$$

by the standard error estimate of Lagrange interpolation, where

$$M_2 = \max_{\xi \in [x_{j-1}, x_j]} |f''(\xi)| .$$

$\pi_2 = (x-x_{j-1})(x-x_j)$ is zero at the boundary of the interval, and has a minimum when $\pi_2'(x) = x - x_j + x - x_{j-1} = 0$

$$\Rightarrow x = \frac{x_{j-1}+x_j}{2}$$

$$\Rightarrow \max_{x\in[x_{j-1},x_j]} |\pi_2(x)| = \left| \underbrace{\left(\frac{x_{j-1}+x_j}{2} - x_{j-1}\right)}_{=\frac{h}{2}} \underbrace{\left(\frac{x_{j-1}+x_j}{2} - x_j\right)}_{=-\frac{h}{2}} \right| = \frac{h^2}{4}$$

Therefore, on the interval $[x_0, x_n]$,

$$|f(x) - S_L(x)| \leq \frac{1}{8} h^2 \max_{\xi\in[x_0,x_n]} |f''(\xi)|$$

2. On each partition, the spline has $n+1$ unknowns. Adding another partition adds 2 interpolation conditions and $n-1$ matching conditions, i.e. the difference between the number of unknowns and the number of conditions remains unchanged.
It is therefore sufficient to consider the spline on a single partition, which must satisfy 2 interpolation condition and no matching condition. We must therefore supply $n-1$ additional conditions to specify the spline uniquely.

3. At node x_i :

$$S_i(x_i) = y_i$$
$$S_{i+1}(x_i) = y_i$$
$$S_i'(x_i) = S_{i+1}'(x_i) = y_i'$$
$$S_i''(x_i) = S_{i+1}''(x_i) = y_i''$$

(a) $S_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$

$$S_i'(x) = 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i$$
$$S_i''(x) = 6a_i(x-x_i) + 2b_i$$

$$S_i(x_i) = y_i \Rightarrow \boxed{d_i = y_i}$$
$$S_i'(x_i) = y_i' \Rightarrow \boxed{c_i = y_i'}$$

$$S_i''(x_i) = S_{i+1}''(x_i) \Rightarrow 2b_i = -6a_{i+1} + 2b_{i+1}$$
$$\Rightarrow 3a_{i+1} = b_{i+1} - b_i$$

$$S_{i+1}(x_i) = y_i \Rightarrow -a_{i+1} + b_{i+1} - c_{i+1} + d_{i+1} = y_i$$

$$\Rightarrow 3a_{i+1} - 3b_{i+1} = -3y'_{i+1} + 3(y_{i+1} - y_i)$$

$$\Rightarrow -2b_{i+1} - b_i = -3y'_{i+1} + 3(y_{i+1} - y_i)$$

$$S'_{i+1}(x_i) = y'_i \Rightarrow 3a_{i+1} - 2b_{i+1} + c_{i+1} = y'_i$$

$$\Rightarrow -b_{i+1} - b_i = y'_i - y'_{i+1}$$

$$\Rightarrow b_{i+1} = 3(y_i - y_{i+1}) + 2y'_{i+1} + y'_i$$

$$\Rightarrow \boxed{b_i = 3(y_{i-1} - y_i) + 2y'_i + y'_{i-1}}$$

$$3a_{i+1} - 2\left[3(y_i - y_{i+1}) + 2y'_{i+1} + y'_i\right] = y'_i - y'_{i+1}$$

$$\Rightarrow a_{i+1} = 2(y_i - y_{i+1}) + y'_{i+1} + y'_i$$

$$\Rightarrow \boxed{a_i = 2(y_{i-1} - y_i) + y'_i + y'_{i-1}}$$

(b) $$y'_i - y'_{i+1} = -b_{i+1} - b_i$$

$$= 3(y_{i+1} - y_i) - 2y'_{i+1} - y'_i + 3(y_i - y_{i-1}) - 2y'_i - y'_{i-1}$$

$$\Rightarrow y'_{i+1} + 4y'_i + y'_{i-1} = 3(y_{i+1} - y_{i-1})$$

Written in matrix form, this equation is as stated.