Homework 5 solutions

1. (a) $$x_{k+1} = x_k + \alpha P^{-1} r_k$$

$$= x_k + \alpha P^{-1}(b - Ax_k)$$

$$= \alpha P^{-1} b + \underbrace{(I - \alpha P^{-1} A)}_{B} x_k$$

We need to find the eigenvalues of B.

<u>Fact:</u> If M has eigenvalue λ, then $I + \alpha M$ has eigenvalue $1 + \alpha\lambda$.

<u>Proof:</u> $$Mv = \lambda v$$
$$\Rightarrow \alpha M v = \alpha \lambda v$$
$$\Rightarrow \underbrace{v + \alpha M v}_{=(I+\alpha M)v} = \underbrace{v + \alpha\lambda v}_{=(1+\alpha\lambda)v}$$ □

Thus, the largest and smallest eigenvalue of B are

$1 - \alpha\lambda_{max}$ and $1 - \alpha\lambda_{min}$,

respectively.

For convergence, we need

$$1 - \alpha\lambda_{max} > -1$$

$$\Rightarrow \alpha < \frac{2}{\lambda_{max}}$$

Since $0 < \lambda_{min} \leq \lambda_{max}$, all other eigenvalues of B are less or equal to $1 - \alpha\lambda_{max}$ in magnitude, and thus of no concern.

(b) The method is optimal, if

$$|\underbrace{1 - \alpha\lambda_{max}}_{<0}| = |\underbrace{1 - \alpha\lambda_{min}}_{>0}|$$

$$\Rightarrow \alpha\lambda_{max} - 1 = 1 - \alpha\lambda_{min}$$

$$\Rightarrow \alpha = \frac{2}{\lambda_{max} + \lambda_{min}}$$ □

2. Let x be a fixed point of $x_{k+1} = x_k + B r_k$

$$\Rightarrow x = x + B(b - Ax)$$

$$\Rightarrow B(b - Ax) = 0$$

since B is invertible, $Ax = b$ □

3. For the Jacobi method,

$$B = I - \begin{pmatrix} a_{11}^{-1} & & 0 \\ & \ddots & \\ 0 & & a_{nn}^{-1} \end{pmatrix} \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & * \\ \vdots & & & \\ a_{n1} & & * & 0 \end{pmatrix}$$

$$\Rightarrow B = \begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 1 & & * \\ \vdots & & \ddots & \\ \frac{a_{n1}}{a_{nn}} & * & & 1 \end{pmatrix}$$

$$\Rightarrow \|B\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |b_{ij}|$$

$$= \max_{i=1,\dots,n} \Bigg[1 + \underbrace{\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|}}_{<1 \quad \text{as} \quad |a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|} \Bigg]$$

$$< 1 .$$

Thus, the Jacobi method converges, as $\rho(B) < \|B\|_\infty < 1$.