Homework 4 Solutions

1. (a) Let λ_{max} be the dominating eigenvalue, i.e $|\lambda_{max}| = \rho(A)$.

There is a corresponding eigenvector $v \neq 0$ s.t

$$Av = \lambda_{max} v$$

$$\Rightarrow \underbrace{\|Av\|}_{\leq \|A\| \|v\|} = \underbrace{\|\lambda_{max} v\|}_{= |\lambda_{max}| \|v\|}$$

$$\Rightarrow \rho(A) \leq \|A\| \qquad \square$$

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has a double eigenvalue $\lambda = 0$, but $A \neq 0$.

2. We split $A = P + (A - P)$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

Then the Jacobi method reads

$$x_{k+1} = P^{-1} b + P^{-1}(P - A) x_k$$

$$= b + \underbrace{\begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}}_{=: B} x_k$$

Eigenvalues of B: $\quad 0 = \det(B - \lambda I) = \lambda^2 - \varepsilon^2$

$$\Rightarrow \lambda = \pm \varepsilon$$

$\Rightarrow$ The spectral radius is < 1, i.e. the method converges, if $|\varepsilon| < 1$.

5. write $Q = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}$ $\qquad R = \begin{pmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{pmatrix}$

$A = \begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{pmatrix}$

The QR decomposition is essentially Gram-Schmidt orthogonalization:

① $q_1 := a_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad$ (q_1 is already normalized)

$r_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad$ check: $\begin{pmatrix} 0 & | & | \\ 1 & q_2 & q_3 \\ 0 & | & | \\ 0 & | & | \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a_1$ ✓

② $\tilde{q}_2 := a_2 - \underbrace{a_2^T q_1}_{=0} q_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\qquad (*)$

$q_2 := \dfrac{\tilde{q}_2}{\|\tilde{q}_2\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

To find r_2, solve $(*)$ for a_2:

$$a_2 = \underbrace{a_2^T q_1}_{=0} q_1 + \underbrace{\|\tilde{q}_2\|}_{=\sqrt{2}} q_2 + 0 \; q_3$$

The coefficients on the right hand side are the components of r_2

$$\Rightarrow r_2 = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

③ $q_3 := a_3 - \underbrace{a_3^T q_1}_{=1} q_1 - \underbrace{a_3^T q_2}_{=\sqrt{2}} q_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$(**)$

$$q_3 := \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

To find r_3, solve (**) for a_3 :

$$a_3 = \underbrace{a_3^T q_1}_{=1} q_1 + \underbrace{a_3^T q_2}_{=\sqrt{2}} q_2 + \underbrace{\|\tilde{q}_3\|}_{=1} q_3$$

$$\Rightarrow \quad r_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad Q = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Least square solution for $Ax = b$:

$$Rx = Q^T b$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{4}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{5}{\sqrt{2}} \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad x_3 = 2$$

$$\sqrt{2}\, x_2 + \sqrt{2}\, x_3 = \frac{5}{\sqrt{2}} \quad \Rightarrow \quad x_2 = \frac{5}{2} - 2 = \frac{1}{2}$$

$$x_1 + 0x_2 + x_3 = 3 \quad \Rightarrow \quad x_1 = 1$$

$$\Rightarrow \quad x = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 2 \end{pmatrix}$$