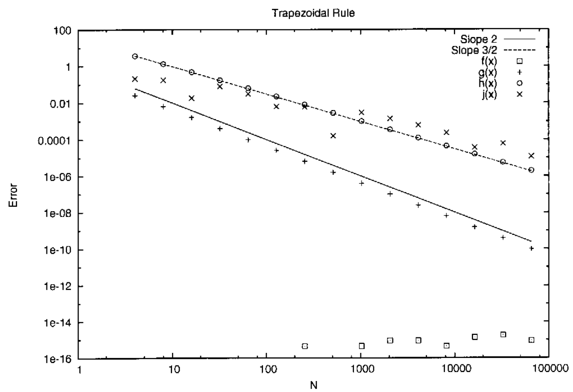
1. You use the composite trapezoidal rule to integrate various functions on the interval $[0, 2\pi]$. The following graph is a log-log plot of the error as a function of the number of partitions.



Match the functions $f(x)$, $g(x)$, $h(x)$, and $j(x)$ to the following expressions and *justify your choice briefly*:

(a) $\sqrt{x} = h(x)$

(b) $x^2 = g(x)$

(c) $(\sin x)^2 = f(x)$

(d) $\begin{cases} 1 & \text{for } x < 1 \\ 0 & \text{otherwise} \end{cases} = j(x)$

(10)

Comments:

(a) h is not differentiable at $x=0$, therefore we expect an order < 2. (In fact, the order is $\frac{3}{2}$, see homework ?)

(b) g is smooth, but non-periodic. The trapezoidal rule shows the generic order 2 behavior.

(c) f is periodic $\Rightarrow$ arbitrary order by Euler-Maclaurin.

(d) j is not even continuous. Order is minimal (i.e. order is 1), but the actual errors fluctuate depending how close the discontinuity is to a quadrature node.

2. (a) Compute the QR decomposition of

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(b) Use the QR decomposition to find the least square solution to $Ax = b$ where

$$b = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

(10+10)

(a) write $Q = \begin{pmatrix} | & | \\ q_1 & q_2 \\ | & | \end{pmatrix}$ $\quad R = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$

• set $\tilde{q}_1 = a_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $\quad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

since $\alpha q_1 = a_1$, $\quad \alpha = \sqrt{3}$.

• set $\tilde{q}_2 = a_2 - \underbrace{\langle a_2, q_1 \rangle}_{= \frac{1}{\sqrt{3}} \cdot 3 = \sqrt{3}} q_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\quad q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

since $\beta q_1 + \gamma q_2 = a_2$, $\quad \beta = \gamma = \sqrt{3}$.

$\Rightarrow Q = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad R = \sqrt{3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(b) $Rx = Q^T b$

$= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \sqrt{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \Rightarrow \boxed{x_2 = 2, \quad x_1 = -1}$

3. (a) Show that you do not need pivoting when computing the LU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 2\varepsilon & 0 \\ \varepsilon & 1 & \varepsilon \\ 0 & 2\varepsilon & 1 \end{pmatrix}$$

whenever $|\varepsilon| < \frac{1}{2}$.

(b) **Extra credit:** Consider the tridiagonal matrix

$$\begin{pmatrix} 1 & a_1 & & & \cdots & 0 \\ b_2 & 1 & a_2 & & & \vdots \\ & b_3 & 1 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & b_{n-1} & 1 & a_{n-1} \\ 0 & \cdots & & & b_n & 1 \end{pmatrix}.$$

Based on your experience from part (a), conjecture and prove a sufficient condition on a_i and b_i for being able to perform LU decomposition without pivoting.

(15 + *10*)

(a) It is easy to find the LU-decomposition:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \varepsilon & 1 & 0 \\ 0 & \frac{2\varepsilon}{1-2\varepsilon^2} & 1 \end{pmatrix}}_{=L} \underbrace{\begin{pmatrix} 1 & 2\varepsilon & 0 \\ 0 & 1-2\varepsilon^2 & \varepsilon \\ 0 & 0 & \frac{1-4\varepsilon^2}{1-2\varepsilon^2} \end{pmatrix}}_{=U}$$

If $|\varepsilon| < \frac{1}{2}$, $1 - 2\varepsilon^2 > \frac{1}{2}$. Therefore

- we are not dividing by a very small number
- since $1 - 4\varepsilon^2 > 0$, the matrix remains nonsingular

(not essential for a successful LU-decomposition, but this shows that the computation could continue without pivoting if there were further rows $\to$ part (b))

(b) An obvious generalization of part (a) is:

If A is tridiagonal and strictly diagonally dominant, then LU-decomposition without pivoting succeeds.

We prove the statement for the indicated special case where all diagonal elements are 1.

It is sufficient to show that one step in the Gaussian elimination process does not destroy strict diagonal dominance; the general statement is obtained by iterating this argument.

$$\begin{pmatrix} 1 & a_1 & 0 & \cdots \\ b_2 & 1 & a_2 & \\ 0 & * & * & \end{pmatrix} \to \begin{pmatrix} 1 & a_1 & 0 & \cdots \\ 0 & 1 & \tilde{a}_2 & 0 \; \cdots \\ 0 & * & * & \end{pmatrix}$$

$$\tilde{a}_2 = \frac{a_2}{1 - a_1 b_2}$$

where $|a_1| < 1$

$|a_2| + |b_2| < 1$

$\Rightarrow |a_2| < 1 - |b_2| < 1 - |a_1||b_2| < |1 - a_1 b_2|$

$\Rightarrow |\tilde{a}_2| < 1$ $\square$

4. (a) Solve the logistic differential equation

$$\dot{y} = y + \mu y^2,$$
$$y(0) = y_0,$$

where $\mu \neq 0$.

(b) Find the critical points, i.e. the zeros of the right hand side of the logistic equation.

(c) Determine the stability of the critical points.
(Note that for a scalar equation $\dot{y} = f(y)$, a critical point y_{crit} is *linearly stable* if $f'(y_{crit}) < 0$, *linearly unstable* if $f'(y_{crit}) > 0$ and *linearly neutrally stable* if $f'(y_{crit}) = 0$).

(10+5+5)

(a) $$\frac{dy}{y(1+\mu y)} = dt$$

$$\Rightarrow \int_{y_0}^{y(t)} \left(\frac{1}{y} - \frac{\mu}{1+\mu y} \right) dy = \int_0^t dt$$

$$\Rightarrow \ln \frac{y}{\frac{1}{\mu}+y} \Bigg|_{y_0}^{y(t)} = t$$

$$\Rightarrow \ln \frac{y(t)}{\frac{1}{\mu}+y(t)} = \ln \underbrace{\frac{y_0}{\frac{1}{\mu}+y_0}}_{=:\,a} + t$$

$$\Rightarrow \frac{y}{\frac{1}{\mu}+y} = a e^t$$

$$\Rightarrow y(t) = \frac{1}{\mu} \frac{a e^t}{1 - a e^t} = \frac{1}{\mu} \frac{\frac{y_0}{\frac{1}{\mu}+y_0}}{e^{-t} - \frac{y_0}{\frac{1}{\mu}+y_0}}$$

$$= \frac{y_0}{(1+\mu y_0)e^{-t} - \mu y_0}$$

(b) $y(1+\mu y) = 0$

$\Rightarrow y = 0$ or $y = -\frac{1}{\mu}$

(c) $f(y) = y + \mu y^2$

$f'(y) = 1 + 2\mu y$

$\Rightarrow f'(0) = 1 > 0$, the critical point at $y=0$ is unstable

$f'(-\frac{1}{\mu}) = 1 + 2\mu(-\frac{1}{\mu}) = -1 < 0$,

the critical point at $y = -\frac{1}{\mu}$ is stable.

5. The following method for solving the differential equation

$$y' = f(y)$$

is called the *trapezoidal rule*:

$$y_{n+1} = y_n + h\frac{f(y_n) + f(y_{n+1})}{2}.$$

(a) Classify the method (implicit or explicit, one-step or multi-step, linear or nonlinear).

(b) Show that you can derive the trapezoidal rule by writing the differential equation in integral form

$$y(t_n + h) = y(t_n) + \int_{t_n}^{t_n+h} f(y(t))\,dt$$

and approximating the integrand by an appropriate Lagrange interpolating polynomial.

(c) Show that the trapezoidal rule is of order 2.
Hint: you may refer to your computation in part (b), or work out the local truncation error.

(d) Show that the region of absolute stability is the left half of the complex λh plane.

(5+5+5+10)

(a) implicit linear one-step

(b)

$$y(t_n+h) \approx y(t_n) + \int_{t_n}^{t_n+h} \left[L_0(t)\, f(y(t_n)) + L_1(t)\, f(y(t_n+h))\right]$$

where $L_0(t) = \dfrac{t-(t_n+h)}{t_n-(t_n+h)}$ $L_1(t) = \dfrac{t-t_n}{(t_n+h)-t_n}$

$$\int_{t_n}^{t_n+h} L_0(t)\,dt = h\int_0^1 (1-t)\,dt \qquad \text{(substitution)}$$

$$= \frac{h}{2}$$

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Similarly $\displaystyle\int_{t_n}^{t_n+h} L_1(t)\,dt = \frac{h}{2}$ (sum needs to be h)

Setting $y(t_n) \to y_n$
$y(t_n+h) \to y_{n+1}$,

we obtain the method

$$y_{n+1} = y_n + \frac{h}{2} f(y_n) + \frac{h}{2} f(y_{n+1})$$

(c) The local truncation error is determined by the interpolation error, bounded by

$$\frac{M_2}{2!}\,\pi_2(t) = O(t^2)$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} \pi_2(t)\,dt = O(h^3)$$

$\Rightarrow$ Global truncation error is $O(h^2)$.

(d) For $y' = \lambda y$, $\quad y_{n+1} = y_n + h\dfrac{\lambda y_n + \lambda y_{n+1}}{2}$

$$\Rightarrow y_{n+1} = \frac{1+\frac{1}{2}\lambda h}{1-\frac{1}{2}\lambda h}\, y_n$$

For absolute stability, we need $\left|\dfrac{1+\frac{1}{2}\lambda h}{1-\frac{1}{2}\lambda h}\right| < 1 \Rightarrow \left|1+\frac{1}{2}\lambda h\right| < \left|1-\frac{1}{2}\lambda h\right|$

$$\Rightarrow \left|1+\tfrac{1}{2}\lambda h\right|^2 < \left|1-\tfrac{1}{2}\lambda h\right|^2$$

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$$\Rightarrow \left(1+\tfrac{1}{2}\lambda h\right)\left(1+\tfrac{1}{2}\bar{\lambda} h\right) < \left(1-\tfrac{1}{2}\lambda h\right)\left(1-\tfrac{1}{2}\bar{\lambda} h\right)$$

$$\Rightarrow 1 + \tfrac{1}{2}(\lambda+\bar{\lambda})h + \tfrac{1}{4}|\lambda|^2 h^2 < 1 - \tfrac{1}{2}(\lambda+\bar{\lambda})h + \tfrac{1}{4}|\lambda|^2 h^2 \Rightarrow \mathrm{Re}(\lambda h) < 0$$

$\square$

6. Consider an implicit one-step method of the form

$$y_{n+1} = y_n + h\,\Phi(y_{n+1})\,,$$

where Φ satisfies a Lipshitz condition, i.e.

$$|\Phi(x) - \Phi(y)| \le L\,|x - y|$$

for all x, y contained in some bounded set D.

(a) Show that simple iteration

$$\eta_{k+1} = g(\eta_k)\,,$$
$$\eta_0 = y_n$$

with $g(\eta) = y_n + h\,\Phi(\eta)$ will converge to y_{n+1} whenever $hL < 1$.
Note: you may assume without further analysis that the iterates never leave D.

(b) **Extra credit:** Would you use simple iteration when applying an implicit method to a stiff system, i.e. to a differential equation that has (in its linearization) a negative eigenvalue of much larger magnitude than all the others? Explain.

(10 + *10*)

(a) $$|g(\eta) - g(\xi)| = |y_n + h\Phi(\eta) - (y_n + h\Phi(\xi))|$$
$$= h\,|\Phi(\eta) - \Phi(\xi)|$$
$$\le hL\,|\eta - \xi|$$

If $hL < 1$, g is a contraction, i.e. simple iteration converges.

(b) No. A stiff system with a large negative eigenvalue λ has $L = |\lambda|$. The step size restriction for an explicit method will be of the form $|h\lambda| < \text{const} \overset{?}{\approx} 1$. Thus, changing to an implicit scheme with convergence criterion $|h\lambda| < 1$ does not improve anything. A different solver (e.g. Newton) is needed.