Numerical Methods I

Problem Set 3

due September 29, 2003

Recall that if $\|\boldsymbol{x}\|_p$ denotes the p-norm of a vector $\boldsymbol{x} \in \mathbb{R}^n$, then the associated norm for a matrix $A \in \mathbb{R}^{n\times n}$ is defined

$$\|A\|_p = \max_{\boldsymbol{x}\neq 0} \frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p}\,.$$

1. (From SM.) Suppose that for a matrix $A \in \mathbb{R}^{n\times n}$,

$$\sum_{i=1}^{n} |a_{ij}| \le C \tag{1}$$

   for $j = 1, \ldots, n$.

   (a) Show that, for any vector $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|A\boldsymbol{x}\|_1 \le C\,\|\boldsymbol{x}\|_1\,. \tag{2}$$

   (b) Find C subject to (1) and a nonzero vector $\boldsymbol{x}$ so that (2) holds with equality.

   (c) Conclude that

$$\|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{n} |a_{ij}|\,.$$

2. (a) Show that, for $A \in \mathbb{R}^{n\times n}$,

$$\|A\|_2 = \sqrt{\lambda_{\max}}\,,$$

   where $\lambda_{\max}$ is the largest eigenvalue of $A^T A$.
   Hint: Recall that for any $\boldsymbol{x} \in \mathbb{R}^n$, you can write $\boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|_2$. The eigenvalues of the symmetric matrix $A^T A$ are real and nonnegative (why?), and its eigenvectors can be chosen to form an orthonormal basis.

   (b) Conclude that the condition number of an orthogonal matrix is 1.

3. (From SM.) Assume that a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ has an LU factorization. Show that A can also be factored in the form $A = LDR$ where L is unit lower triangular, D is diagonal, and R is unit upper triangular. Use this result to express the LU factoriztion of A^T in terms of the LU factorization of A.

4. The $n \times n$ Vandermonde Matrix is defined as

$$V = \begin{pmatrix} 1 & 2 & 4 & 8 & \cdots & 2^{n-1} \\ 1 & 3 & 9 & 27 & \cdots & 3^{n-1} \\ 1 & 4 & 16 & 64 & \cdots & 4^{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & n+1 & (n+1)^2 & (n+1)^3 & \cdots & (n+1)^{n-1} \end{pmatrix}.$$

Let $\boldsymbol{b} \in \mathbb{R}^n$ be the vector containing the sum of each row of V. Find a formula for the components of $\boldsymbol{b}$. What is the solution of the system of linear equations $V\boldsymbol{x} = \boldsymbol{b}$?

5. **Project:** Write an Octave function `vandermonde(n)` that generates the $n \times n$ Vandermonde Matrix. Compute its condition number with respect to the 2-norm for different values of n. (You may use the built-in Octave function `cond`.)

6. **Project:** Write an Octave function that computes the LU decomposition of a matrix without pivoting. Test your code by solving $V\boldsymbol{x} = \boldsymbol{b}$ from Question 4 for several values of n.