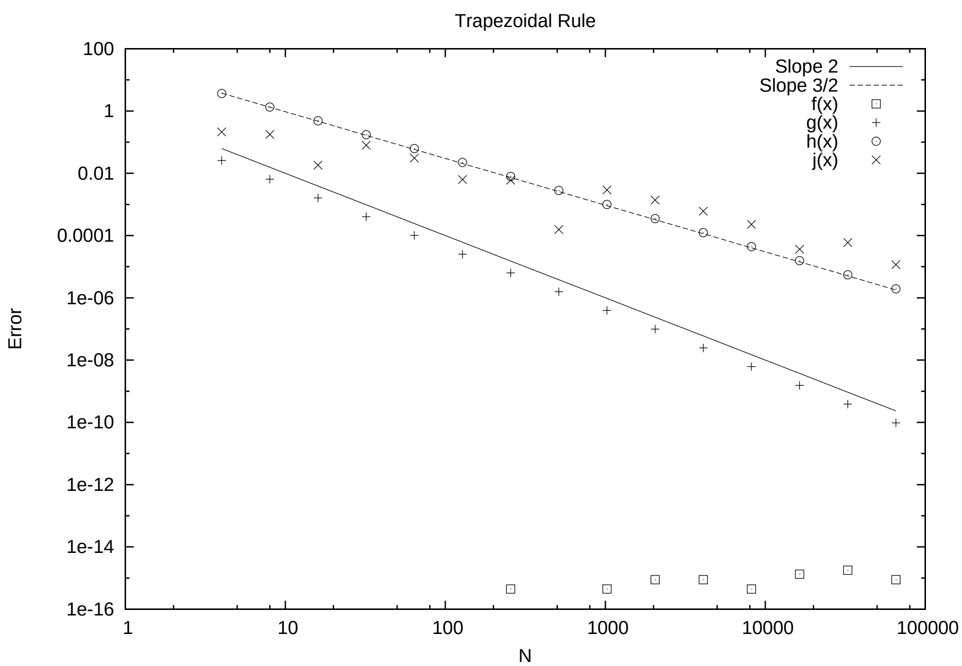
Numerical Methods I

Final Exam

December 12, 2003

1. You use the composite trapezoidal rule to integrate various functions on the interval $[0, 2\pi]$. The following graph is a log-log plot of the error as a function of the number of partitions.



Match the functions $f(x)$, $g(x)$, $h(x)$, and $j(x)$ to the following expressions and *justify your choice briefly*:

(a) $\sqrt{x}$

(b) x^2

(c) $(\sin x)^2$

(d) $\begin{cases} 1 & \text{for } x < 1 \\ 0 & \text{otherwise} \end{cases}$

(10)

2. (a) Compute the QR decomposition of

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} .$$

(b) Use the QR decomposition to find the least square solution to $A\boldsymbol{x} = \boldsymbol{b}$ where

$$\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix} .$$

(10+10)

3. (a) Show that you do not need pivoting when computing the LU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 2\varepsilon & 0 \\ \varepsilon & 1 & \varepsilon \\ 0 & 2\varepsilon & 1 \end{pmatrix}$$

whenever $|\varepsilon| < \frac{1}{2}$.

(b) **Extra credit:** Consider the tridiagonal matrix

$$\begin{pmatrix} 1 & a_1 & & & \cdots & 0 \\ b_2 & 1 & a_2 & & & \vdots \\ & b_3 & 1 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & b_{n-1} & 1 & a_{n-1} \\ 0 & \cdots & & & b_n & 1 \end{pmatrix} .$$

Based on your experience from part (a), conjecture and prove a sufficient condition on a_i and b_i for being able to perform LU decomposition without pivoting.

(15+*10*)

4. (a) Solve the logistic differential equation

$$\dot{y} = \mu\, y + y^2 \,, \qquad y(0) = y_0 \,,$$

where $\mu \neq 0$.

(b) Find the critical points, i.e. the zeros of the right hand side of the logistic equation.

(c) Determine the stability of the critical points.

(Note that for a scalar equation $\dot{y} = f(y)$, a critical point y_{crit} is *linearly stable* if $f'(y_{\text{crit}}) < 0$, *linearly unstable* if $f'(y_{\text{crit}}) > 0$ and *linearly neutrally stable* if $f'(y_{\text{crit}}) = 0$).

(10+5+5)

5. The following method for solving the differential equation

$$y' = f(y)$$

is called the *trapezoidal rule*:

$$y_{n+1} = y_n + h\,\frac{f(y_n) + f(y_{n+1})}{2}\,.$$

(a) Classify the method (implicit or explicit, one-step or multi-step, linear or nonlinear).

(b) Show that you can derive the trapezoidal rule by writing the differential equation in integral form

$$y(t_n + h) = y(t_n) + \int_{t_n}^{t_n+h} f(y(t))\,\mathrm{d}t$$

and approximating the integrand by an appropriate Lagrange interpolating polynomial.

(c) Show that the trapezoidal rule is of order 2.

Hint: you may refer to your computation in part (b), or work out the local truncation error.

(d) Show that the region of absolute stability is the left half of the complex λh plane.

(5+5+5+10)

6. Consider an implicit one-step method of the form

$$y_{n+1} = y_n + h\,\Phi(y_{n+1})\,,$$

where Φ satisfies a Lipshitz condition, i.e.

$$|\Phi(x) - \Phi(y)| \le L\,|x - y|$$

for all x, y contained in some bounded set D.

(a) Show that simple iteration

$$\eta_{k+1} = g(\eta_k)\,,$$
$$\eta_0 = y_n$$

with $g(\eta) = y_n + h\,\Phi(\eta)$ will converge to y_{n+1} whenever $hL < 1$.

Note: you may assume without further analysis that the iterates never leave D.

(b) **Extra credit:** Would you use simple iteration when applying an implicit method to a stiff system, i.e. to a differential equation that has (in its linearization) a negative eigenvalue of much larger magnitude than all the others? Explain.

(10+*10*)