Partial Differential Equations

Homework 7

due December 2, 2002

In the following, $\mathbb{T}$ denotes the 1-torus, i.e. $\mathbb{T} = \mathbb{R} \bmod 2\pi$.

1. (a) Show that, for every $u \in H^2(\mathbb{T})$,

$$\|u\|_{H^1}^2 \leq \|u\|_{L^2} \|u\|_{H^2} .$$

   (b) Consider the Fisher–Kolmogorov equation on $\mathbb{T}$,

$$u_t = u_{xx} + (1-u)\, u^m ,$$
$$u(0) = u^{\text{in}} ,$$

   where m is an even positive integer. Use the result from (a), as well as the first question of the previous homework set, to prove that that

$$\limsup_{t \to \infty} \|u(t)\|_{H^1} \leq C$$

   where an explicit estimate for C can be given which, in particular, shows that C does not depend on the initial data u^{in}. You may assume that u is sufficiently differentiable so that all your formal manipulations are justified.

2. Prove the following version of the *Poincaré inequality*: For every $u \in H^1(\mathbb{T})$ which has zero mean, i.e. where

$$⨍_{\mathbb{T}} u \, dx = 0 ,$$

   we have

$$\int_{\mathbb{T}} |u|^2 \, dx \leq C \int_{\mathbb{T}} |u_x|^2 \, dx .$$

   Find the best estimate for C.