Partial Differential Equations

Homework 6

due November 13, 2002

In the following, $\mathbb{T}$ denotes the 1-torus, i.e. $\mathbb{T} = \mathbb{R} \bmod 2\pi$.

1. (a) Show that, for every $u \in L^r(\mathbb{T})$ with $2 \le r < \infty$,

$$\|u\|_{L^2} \le (2\pi)^{\frac{r-2}{2r}} \|u\|_{L^r} .$$

Hint: Hölder inequality.

(b) Consider the Fisher–Kolmogorov equation on $\mathbb{T}$,

$$u_t = u_{xx} + (1-u)\,u^m ,$$
$$u(0) = u^{\text{in}} ,$$

where m is an even positive integer. Use the result from (a) to sharpen the L^2 estimate derived in the lecture as follows: Show that

$$\limsup_{t\to\infty} \|u(t)\|_{L^2} \le C$$

where an explicit estimate for C can be given which, in particular, shows that C does not depend on the initial data u^{in}.

2. Show that if $u^{\text{in}} \ge 0$, the solution $u(t)$ to the Fisher–Kolmogorov equation remains nonnegative for every $t \ge 0$. You may assume that u is as smooth as you need.

Hint: This is similar to question 1 of the previous homework.

3. Let $\{u_n\} \subset L^2(U)$ be a weakly convergent sequence. Show that $\{u_n\}$ is bounded.

4. (a) Prove the following elementary version of the *Rellich Theorem*:
The embedding $H^t(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$ is compact for all real numbers $s < t$.
This is equivalent to saying that if $u_n \rightharpoonup u$ weakly in $H^t(\mathbb{T})$, then $u_n \to u$ strongly in $H^s(\mathbb{T})$.
Hints: WLOG $u = 0$; use the Fourier series representation of the H^s norms.

(b) Use the result from part (a) to show that for every $T > 0$ the embedding

$$L^2([0,T]; L^2(\mathbb{T})) \hookleftarrow C([0,T]; \text{w-}L^2(\mathbb{T})) \cap \text{w-}L^2([0,T]; \text{w-}H^1(\mathbb{T})) ,$$

is compact, where the intersection on the right side is endowed with the relative topology induced by the inclusion map. In other words, a sequence converges in the intersection iff it converges in each of the spaces separately.