Partial Differential Equations

Homework 2

due September 25, 2002

1. Verify the prefactors in the definition of the averages

$$\fint_{B(x,r)} u(y)\,dy \equiv \frac{1}{\alpha(n)\,r^n} \int_{B(x,r)} u(y)\,dy$$

and

$$\fint_{\partial B(x,r)} u(y)\,dy \equiv \frac{1}{n\,\alpha(n)\,r^{n-1}} \int_{\partial B(x,r)} u(y)\,dy\,.$$

2. Let $X \subset \mathbb{R}^n$. Show that

   (a) X is connected iff $\emptyset$ and X are the only subsets of X that are both relatively open and relatively closed in X.

   (b) If $\{W_\alpha\}_{\alpha\in A}$ is a collection of connected subsets of X such that

$$\bigcap_{\alpha\in A} W_\alpha \neq \emptyset\,,$$

   then $\cup_{\alpha\in A} W_\alpha$ is connected.

   (c) If X is connected, then $\overline{X}$ is connected.

   (d) Every point $x \in X$ is contained in a unique maximal connected subset of X, and this subset is relatively closed in X.

   The relevant definitions from point-set topology in $\mathbb{R}^n$:

   - $A \subset X$ is called *relatively open in* X if for every $x \in A$ there exists an $\varepsilon > 0$ such that $X \cap B(x,\varepsilon) \subset A$.
   - $B \subset X$ is called *relatively closed in* X if $A = X \setminus B$ is relatively open in X.
   - X is called *disconnected* if there exist disjoint, nonempty subsets $A_1, A_2 \subset X$ that are relatively open in X and $X = A_1 \cup A_2$.
   - X is called *connected* if it is not disconnected.

3. Evans, p. 85/86 problems 3 and 4.