# Partial Differential Equations

Final Exam

December 19, 2002

**Get 50 out of 70 points for a 100% score.**

1. (a) Show that the Fourier transform for $f \colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = e^{-t|x|}\,,$$

is given by

$$\hat{f}(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{t}{\xi^2 + t^2}\,.$$

(b) Use the result from (a) to solve the 1-dimensional differential equation

$$u_{xx} - u = \delta\,,$$

where $\delta$ is the Dirac measure centered at $x = 0$.

(5+5)

2. Assume that $u \in C^3(\mathbb{R}^n \times [0, \infty))$ solves the heat equation

$$u_t = \Delta u\,.$$

Prove that $v \equiv |Du|^2 + u_t^2$ is a *subsolution* of the heat equation, i.e.

$$v_t \leq \Delta v\,.$$

(10)

3. Find an *a priori* estimate which can be used to prove that if $u \in H^1(\mathbb{R})$, then $u$ is continuous.

Hint: Use the Fundamental Theorem of Calculus to estimate $|u(x) - u(y)|$. (10)

4. Show that

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} u_t^2\,dx + \frac{1}{2}\int_{\mathbb{R}} u_x^2\,dx$$

is a constant in time for smooth solutions $u \colon \mathbb{R} \times [0, \infty) \to \mathbb{R}$ *with compact support in space* of the wave equation

$$u_{tt} - u_{xx} = 0\,.$$

(10)

5. A function $G_L$ is called the Green's Function of a linear partial differential operator $L$ if

$$L\,G_L = \delta\,,$$

where $\delta$ is the Dirac measure centered at zero. Show, given two partial linear differential operators $A$ and $B$ with $A + B = 1$, that

$$G_{AB} = G_A + G_B\,.$$

Note: You may assume without further discussion that expressions like $A\delta$ and $B\delta$ can be made sense of. (10)

6. Consider the initial value problem for Burger's equation

$$u_t + u\,u_x = 0\,,$$
$$u(0) = u^{\text{in}}\,,$$

where $u\colon \mathbb{R} \times [0, \infty) \to \mathbb{R}$, and we write $u(t)$ as shorthand for $u(\,\cdot\,, t)$.

(a) Show that, provided $u \in C^1(\mathbb{R} \times [0, \infty))$ and $u^{\text{in}} \in L^2(\mathbb{R})$,

$$\|u(t)\|_{L^2} = \|u^{\text{in}}\|_{L^2}\,. \tag{*}$$

(b) Show that, provided $u \in C^1(\mathbb{R} \times [0, \infty))$,

$$\int_0^\infty \int_{\mathbb{R}} \left(\phi_t\,u + \tfrac{1}{2}\,\phi_x\,u^2\right) dx\,dt = -\int_{\mathbb{R}} \phi(x, 0)\,u^{\text{in}}(x)\,dx \tag{**}$$

for every test function $\phi \in C^1(\mathbb{R} \times [0, \infty))$ with compact support.

(c) It is known that for $0 \leq t \leq 2$ the following function is a so-called *entropy solution* of Burger's equation:

$$u(x, t) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{x}{t} & \text{for } 0 < x < t \\ 1 & \text{for } t \leq x < 1 + \frac{t}{2} \\ 0 & \text{for } x \geq 1 + \frac{t}{2} \end{cases}$$

Show, by direct calculation, that $u$ does *not* satisfy (*).

(d) Show, again by direct calculation, that $u$ as defined in (c) satisfies (**).

(5+5+5+5)