WEAK SOLUTIONS FOR GENERALIZED LARGE-SCALE SEMIGEOSTROPHIC EQUATIONS

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Abstract. We prove existence, uniqueness and continuous dependence on initial data of global weak solutions to the generalized large-scale semigeostrophic equations with periodic boundary conditions. This family of Hamiltonian balance models for rapidly rotating shallow water includes the L_1 model derived by R. Salmon in 1985 and its 2006 generalization by the second author. The analysis is based on the vorticity formulation of the models supplemented by a nonlinear velocity-vorticity relation. The results are fundamentally due to the conservation of potential vorticity. While classical solutions are known to exist provided the initial potential vorticity is positive—a condition which is already implicit in the formal derivation of balance models, we can assert the existence of weak solutions only under the slightly stronger assumption that the potential vorticity is bounded below by $\sqrt{5}-2$ times the equilibrium potential vorticity. The reason is that the nonlinearities in the potential vorticity inversion are felt more strongly when working in weaker function spaces. Another manifestation of this effect is that point-vortex solutions are not supported by the model even in the special case when the potential vorticity inversion gains three derivatives in spaces of classical functions.

1. Introduction

The generalized large-scale semigeostrophic (gLSG) equations belong to the class of fluid equations in two spatial dimensions which can be formulated as an advection equation for a scalar potential vorticity (PV) $q = q(\boldsymbol{x}, t)$ by a two dimensional velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$,

$$\partial_t q + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0 \,, \tag{1a}$$

where the velocity field is slaved to the potential vorticity by some vorticity inversion law which we abbreviate as

$$\boldsymbol{u} = \boldsymbol{K}(q) \,. \tag{1b}$$

This structure is typical for two-dimensional fluid flow. For example, the incompressible Euler equations arise when $\boldsymbol{K}$ represents the classical Biot–Savart law. The well-established mathematical theory for the Euler equations persists when the Biot–Savart law is replaced by a general linear vorticity inversion that gains one derivative in Sobolev space; see [11]. Another well-understood example are the Euler-α (or Lagrangian averaged Euler) equations which were, in their inviscid form, first derived by Holm *et al.* [9] on formal grounds and later justified under certain closure assumptions in [8, 12]. Here, the PV inversion gains three derivatives

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in Sobolev space. For the gLSG equations, the vorticity inversion $\boldsymbol{K}$ is implicitly defined via the system of—as we shall show—elliptic equations

$$(q - \sigma\Delta)h = 1\,, \tag{2a}$$

$$\big(1 - \sigma\,(h\,\Delta + 2\,\boldsymbol{\nabla} h \cdot \boldsymbol{\nabla})\big)\boldsymbol{u} = \boldsymbol{\nabla}^\perp\big(h - \mu\,(2\,h\,\Delta h + |\boldsymbol{\nabla} h|^2)\big)\,. \tag{2b}$$

Here, $\sigma > 0$ and μ are real parameters, $h = h(\boldsymbol{x}, t)$ is a scalar field, and we write $\boldsymbol{\nabla}^\perp = (-\partial_2, \partial_1)$. When $\mu = \sigma/2$, the system is known as the L_1 model due to Salmon [17]; in this case σ is the Rossby number of the flow which is less than one for rotation-dominated flows such as the large-scale motion of atmosphere and ocean in the mid-latitudes. Subsequent work [13] generalized Salmon's approach, interpreting different choices of σ and μ as approximate near-identity changes of the coordinate frame; the case of spatially varying Coriolis parameter and nontrivial bottom topography is discussed in [16].

A naive count of orders of differentiation in (2) shows that, in general, $\boldsymbol{K}$ is expected to gain one derivative in Sobolev space, akin to the Euler equations. In the special case when $\mu = 0$, the relation is expected to gain three derivatives, akin to the Euler-α equations. We note, however, that $\boldsymbol{K}$ is nonlinear so that it is a nontrivial question whether well-known analytical results for Euler or Euler-α continue to hold in this setting. In many respects, these analogies are indeed true. Namely, under the condition that the initial PV is positive, a restriction which is consistent with the physical assumptions underlying the derivation of these balance models, the gLSG equations are Hamiltonian and have local classical solutions [15]; existence of global classical solutions was proved in [5].

In this paper, we address the question of weak solutions under the simplifying assumption of periodic boundary conditions. We find that weak solutions in the sense of Yudovich are borderline independent of whether or not $\mu = 0$; they already incur an additional restriction on the data which is neither seen for linear vorticity relations, nor in the theory of classical gLSG solutions. Radon-measured potential vorticities as can be shown to make sense in the Euler-α case [14], however, appear to break down altogether here.

A weak solution to the gLSG equations is a function $q \in C([0,\infty); \text{w}^*\text{-}L^\infty(\mathbb{T}^2))$ satisfying

$$\langle \psi, q(t_2) \rangle - \langle \psi, q(t_1) \rangle - \int_{t_1}^{t_2} \langle \boldsymbol{\nabla} \cdot (\psi \boldsymbol{u}), q \rangle \, \mathrm{d}t = 0\,, \tag{3a}$$

$$\boldsymbol{u} = \boldsymbol{K}(q)\,, \tag{3b}$$

$$q(0) = q^{\mathrm{in}} \tag{3c}$$

for every $[t_1, t_2] \subset [0, \infty)$ and every test function $\psi \in H^1(\mathbb{T}^2)$.

To simplify notation, we rescale the equations such that $q = h = 1$ is the trivial equilibrium solution. Namely, define, for $f \in L^\infty(\mathbb{T}^2)$,

$$f_- = \operatorname*{ess\,inf}_{\boldsymbol{x} \in \mathbb{T}^2} f(\boldsymbol{x}) \qquad \text{and} \qquad f_+ = \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathbb{T}^2} f(\boldsymbol{x})\,. \tag{4}$$

Then, letting $Q = (q_- + q_+)/2$, we replace q by q/Q, ε by ε/Q, h by hQ, $\boldsymbol{u}$ by $\boldsymbol{u}Q$, and t by t/Q, and note that the generalized LSG equations are invariant under this rescaling. We conclude that, without loss of generality, we may assume that $q_+ - 1 = 1 - q_-$ and write $q \equiv 1 + \tilde{q}$ with $\|\tilde{q}\|_{L^\infty} < 1$. Our main result can then be stated as follows.

Theorem 1. *For initial potential vorticity $q^{\text{in}} \in L^\infty$ with $\|\tilde{q}^{\text{in}}\|_{L^\infty} < \sqrt{5} - 2$, there exists a unique global weak solution of the generalized LSG equations*

$$q \in C([0,\infty); \mathrm{w}^*\text{-}L^\infty(\mathbb{T}^2)) \cap L^\infty([0,\infty) \times \mathbb{T}^2) \tag{5}$$

such that

$$3 - \sqrt{5} < q_-^{\text{in}} \leq q(t) \leq q_+^{\text{in}} < \sqrt{5} - 1 \tag{6}$$

for all $t \in [0,\infty)$. Furthermore, the solution map $q^{\text{in}} \mapsto q(t)$ is continuous in the H^{-1}-topology for every fixed $t \in [0,\infty)$.

Setting and proof follow the construction of weak solutions to the incompressible Euler equations in two dimensions by Yudovich [19] and its later generalization to arbitrary first order linear vorticity inversion operators and weighted divergence condition in [11]. We remark that Bardos [3] proved similar results for the Euler equations in their velocity formulation, also for inhomogeneous boundary conditions.

The problem considered here differs from these earlier works in two respects: our velocity field $\boldsymbol{u}$ is not divergence free and the potential vorticity inversion is nonlinear. In particular, uniform continuity of $\boldsymbol{K}$ has to be proved explicitly as it cannot be inferred from boundedness of the operator. When establishing existence of classical solutions [5], we were working in spaces of sufficiently regular functions where estimates on $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ essentially came for free. This is no longer the case here. Fortunately, taking the divergence of (2b), the entire right hand side which contains the most singular terms drops out so that we can prove estimates on $\boldsymbol{\nabla} \cdot \boldsymbol{K}$ which are almost as strong as those for $\boldsymbol{K}$. However, we can only assert that $\boldsymbol{\nabla} \cdot \boldsymbol{K}$ is uniformly continuous as an operator from $W^{-1,p}$ into L^p provided that $q_- > \sqrt{5} - 2$, hence the restriction on the initial PV in Theorem 1.

Most of the difficulty comes from the fact that the nonlinearities in the potential vorticity inversion are felt more strongly when working in weaker spaces—the estimates used to establish existence of global classical solutions [5] do not simply translate down the scale of Sobolev spaces to the spaces of distributions $W^{-m,p}$ in which we can assert compactness of approximating sequences in the functional setting of Theorem 1 and which must also be used to estimate the right hand side of (2b).

In the classical setting, it is often easiest to follow an “artificial viscosity” approach where the regularized system can be shown to possess L^∞ bounds which remain uniform in the zero viscosity limit. While this construction would be possible here as well, we choose to take advantage of the already established existence of classical solution. Hence, we construct the solution to the weak formulation as a limit of classical solutions corresponding to a sequence of mollifications of our L^∞ initial potential vorticity.

The remainder of the article is structured as follows. In the following section, we briefly establish notational conventions. The main new work lies in Section 3 where we prove the kinematic estimates that characterize our potential vorticity inversion and its divergence. In Section 4, we look at the time-dependent problem, essentially along the lines of the classical theory. Section 5 concludes with a brief remark on the impossibility of point-vortex solutions for gLSG.

While we do not have proof that the $\sqrt{5} - 2$ barrier is sharp or even necessary, we do not see a way to improve the result with current techniques. Modulo potential marginal improvements, we believe that this result is as good as possible regarding generalized or weak solutions to the gLSG equations.

2. Notation and preliminaries

For $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$, we write $W^{m,p}(\mathbb{T}^2)$ to denote the Sobolev space of Lebesgue measurable functions whose weak derivatives up to order m belong to $L^p(\mathbb{T}^2)$, endowed with norm

$$\|f\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha f\|_{L^p} \,, \tag{7}$$

where we employ the usual multi-index notation. We abbreviate $H^m = W^{m,2}$; it is a Hilbert space with inner product

$$\langle f, g \rangle_m = \sum_{|\alpha| \leq m} \langle \mathrm{D}^\alpha f, \mathrm{D}^\alpha g \rangle_{L^2} \,. \tag{8}$$

For convenience, $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_0$ is used to denote the L^2 inner product.

For $m \in \mathbb{N}$ and $p \in (1, \infty)$ with Hölder conjugate $p' = p/(p-1)$, we set $W^{-m,p}(\mathbb{T}^2) = W^{m,p'}(\mathbb{T}^2)'$, endowed with the dual norm

$$\|f\|_{W^{-m,p}} = \sup_{\substack{\phi \in W^{m,p'} \\ \phi \neq 0}} \frac{\langle \phi, f \rangle_{W^{m,p'}, W^{-m,p}}}{\|\phi\|_{W^{m,p'}}} \,. \tag{9}$$

We remark that this definition coincides with the usual definition of $W^{-m,p}$ as the dual of $W_0^{m,p'}$ (see, e.g., [2]), because on the torus the spaces $W^{m,p'}$ and $W_0^{m,p'}$ coincide.

We adopt the following convention on the naming of constants. Constants which might depend on parameters only are denoted by c, constants which may also depend on the data or the bound r to be introduced below are denoted C. Different subscripts indicate a change in the constant from step to step within a single computation; however, we make no attempt at a unique naming of constants across different sections of the paper.

Recall that elliptic L^p theory [4, 7, 18] implies that

$$(1 - \sigma\Delta) \colon W^{2,p}(\mathbb{T}^2) \to L^p(\mathbb{T}^2) \tag{10}$$

is an isomorphism for every $p \in (1, \infty)$ and that there exists a constant c independent of p and σ such that the norm of its inverse is bounded by

$$\|(1 - \sigma\Delta)^{-1}\|_{L^p \to W^{2,p}} \leq \frac{c}{\sigma} \, \frac{p^2}{p-1} \,. \tag{11}$$

This estimate directly translates down the Sobolev scale into a statement on the Helmholtz equation with distribution-valued data; for details see, e.g., [5].

Lemma 2. *Suppose $f \in W^{-2,p}(\mathbb{T}^2)$ with $p \in (1, \infty)$. Then the Helmholtz equation*

$$(1 - \sigma\Delta) v = f \tag{12}$$

has a unique weak solution $v \in L^p(\mathbb{T}^2)$ and there exists a constant c independent of $p \geq 2$ and σ such that

$$\|v\|_{L^p} \leq \frac{c\,p}{\sigma} \, \|f\|_{W^{-2,p}} \,. \tag{13}$$

If, moreover, $f \in W^{-1,p}(\mathbb{T}^2)$*, then* $v \in W^{1,p}(\mathbb{T}^2)$ *with*

$$\|v\|_{W^{1,p}} \leq \frac{c\,p}{\sigma} \, \|f\|_{W^{-1,p}} \, . \tag{14}$$

3. Kinematic estimates

In this section we establish sufficient conditions under which the operators $\boldsymbol{K}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{K}$ are well-defined, and derive kinematic estimates for later use. This task naturally splits into three parts: we first study the second order differential operator from (2a), which we abbreviate

$$L_q \equiv q - \sigma \Delta \, . \tag{15}$$

We then look at the second order differential operator on the left hand side of (2b), which we abbreviate

$$\Lambda_h \equiv 1 - \sigma \, (h \, \Delta + 2 \, \boldsymbol{\nabla} h \cdot \boldsymbol{\nabla}) \, , \tag{16}$$

under the assumption that h satisfies equation (2a). Finally, we estimate the terms on the right of (2b), thereby completing the estimates for the full potential vorticity inversion.

The characterization of L_q is given in the following two propositions.

Proposition 3. *Suppose* $\tilde{q} \in L^\infty(\mathbb{T}^2)$ *with* $\|\tilde{q}\|_{L^\infty} \leq r < 1$*. Then* L_q *has a continuous inverse as an operator from* $W^{2,p}(\mathbb{T}^2)$ *to* $L^p(\mathbb{T}^2)$*, from* $W^{1,p}(\mathbb{T}^2)$ *to* $W^{-1,p}(\mathbb{T}^2)$*, and from* $L^p(\mathbb{T}^2)$ *to* $W^{-2,p}(\mathbb{T}^2)$ *for every* $p \in (1,\infty)$*. The norms of the inverse depend on* r*,* p*, and* σ*, but not otherwise on* q*. Specifically, when*

$$L_q h = f \, , \tag{17}$$

then

$$\|h\|_{L^p} \leq \frac{1}{1-r} \, \|f\|_{L^p} \tag{18a}$$

and there exists a constant c *independent of* r*,* p*, and* σ *such that*

$$\|h\|_{W^{2,p}} \leq \frac{c}{\sigma} \, \frac{p^2}{p-1} \, \frac{1}{1-r} \, \|f\|_{L^p} \, . \tag{18b}$$

Finally, if $r \in [0,1)$ *and* $f \in W^{-1,s}$ *with* $s > 2$ *are fixed, then* $q \mapsto L_q^{-1} f$ *is uniformly continuous on the set* $\{q = 1 + \tilde{q} \colon \|\tilde{q}\|_{L^\infty} \leq r\}$ *as a map from* $W^{-1,p}(\mathbb{T}^2)$ *to* $W^{1,p}(\mathbb{T}^2)$ *for every* $2 \leq p \leq s$*. Specifically, there exists a constant* C *depending on all parameters as well as* r *and* $\|f\|_{W^{-1,s}}$ *such that*

$$\|L_{q_2}^{-1} f - L_{q_1}^{-1} f\|_{W^{1,p}} \leq C \, \|q_2 - q_1\|_{W^{-1,p}} \tag{19}$$

so long as $\|\tilde{q}_i\|_{L^\infty} \leq r$ *for* $i = 1,2$*.*

Proof. Since $q \geq 1 - r > 0$ a.e., the second order operator L_q is uniformly elliptic and the associated bilinear form is coercive. Hence, invertibility as an operator from $W^{2,p}$ to L^p follows by standard elliptic L^p theory. To proceed, we write (17) in fixed point form, namely

$$h = (1 - \sigma\Delta)^{-1} (f - \tilde{q} \, h) \, . \tag{20}$$

Hence,

$$\|h\|_{L^p} \leq \|(1-\sigma\Delta)^{-1}\|_{L^p \to L^p} \left(\|f\|_{L^p} + \|\tilde{q}\|_{L^\infty} \, \|h\|_{L^p} \right) . \tag{21}$$

Since the inverse Helmholtz operator has unit norm on L^p, as can be seen from its explicit integral representation, estimate (18a) is immediate. Moreover, considering the inverse Helmholtz operator as a map from L^p to $W^{2,p}$ in (21) and recalling its norm from (11) yields (18b). Further, we obtain from (20) that

$$\|h\|_{L^p} \leq \|(1-\sigma\Delta)^{-1}\|_{W^{-2,p}\to L^p} \, \|f\|_{W^{-2,p}} + \|(1-\sigma\Delta)^{-1}\|_{L^p\to L^p} \, \|\tilde{q}\|_{L^\infty} \|h\|_{L^p} \tag{22}$$

so that

$$\|h\|_{L^p} \leq \frac{c\,p}{\sigma} \, \frac{1}{1-r} \, \|f\|_{W^{-2,p}} \,. \tag{23}$$

Finally, to obtain a $W^{1,p}$ bound on h, take the $W^{1,p}$ norm of (20) and consider the inverse Helmholtz operator as a map from $W^{-1,p}$ to $W^{1,p}$. Due to (23),

$$\|\tilde{q}h\|_{W^{-1,p}} \leq \|\tilde{q}h\|_{L^p} \leq \frac{c\,p}{\sigma} \, \frac{1}{1-r} \, \|\tilde{q}\|_{L^\infty} \, \|f\|_{W^{-2,p}} \,, \tag{24}$$

so that

$$\|h\|_{W^{1,p}} \leq C(p,r,\sigma) \, \|f\|_{W^{-1,p}} \,. \tag{25}$$

Then (23) and (25) imply that L_q^{-1} extends continuously to a map from $W^{-2,p}$ to L^p and from $W^{-1,p}$ to $W^{1,p}$ as claimed.

To prove uniform continuity, suppose $L_{q_i} h_i = f$ with $\|\tilde{q}_i\|_{L^\infty} < r$ for $i = 1,2$. Then

$$h_2 - h_1 = L_{q_2}^{-1}[h_1(q_1 - q_2)] \tag{26}$$

so that, due to (25),

$$\|h_2 - h_1\|_{W^{1,p}} \leq C(p,r,\sigma) \, \|h_1(q_2 - q_1)\|_{W^{-1,p}} \,. \tag{27}$$

To complete the proof of (19), we must show that

$$\|h_1(q_2 - q_1)\|_{W^{-1,p}} \leq C \, \|q_2 - q_1\|_{W^{-1,p}} \,. \tag{28}$$

First, if $h_1 = 0$ or $h_1(q_2 - q_1) = 0$, this claim is trivial, so assume otherwise. Second, for $\phi \in W^{1,p'}$,

$$\begin{aligned} \|\phi h_1\|_{W^{1,p'}} &\leq \|\phi\|_{L^{p'}} \, \|h_1\|_{L^\infty} + \|\boldsymbol{\nabla}\phi\|_{L^{p'}} \, \|h_1\|_{L^\infty} + \|\phi\|_{L^t} \, \|\boldsymbol{\nabla} h_1\|_{L^s} \\ &\leq c(p,s) \, \|\phi\|_{W^{1,p'}} \, \|h_1\|_{W^{1,s}} \\ &\leq C(s,\sigma,r) \, \|\phi\|_{W^{1,p'}} \, \|f\|_{W^{-1,s}} \,, \end{aligned} \tag{29}$$

where, in the first inequality, we use the Hölder inequality with $1/t + 1/s = 1/p'$ and, in the second inequality, we note the continuity of the embeddings $W^{1,s} \hookrightarrow L^\infty$ and $W^{1,p'} \hookrightarrow L^t$; the last step in (29) is due to (25). Hence,

$$\begin{aligned} \|h_1(q_2 - q_1)\|_{W^{-1,p}} &= \sup_{\substack{\phi \in W^{1,p'} \\ \phi \neq 0}} \frac{\langle \phi, h_1(q_2 - q_1) \rangle}{\|\phi\|_{W^{1,p'}}} \\ &\leq C \sup_{\substack{\phi h_1 \in W^{1,p'} \\ \phi h_1 \neq 0}} \frac{\langle \phi h_1, q_2 - q_1 \rangle}{\|\phi h_1\|_{W^{1,p'}}} = C \, \|q_2 - q_1\|_{W^{-1,p}} \,. \end{aligned} \tag{30}$$

This completes the proof. □

Proposition 3 implies, in particular, that for any function $q \in L^\infty(\mathbb{T}^2)$ with $\|\tilde{q}\|_{L^\infty} < 1$ the solution to $L_q h = 1$ satisfies $h \in W^{2,p}(\mathbb{T}^2)$ for all $1 < p < \infty$. For future reference, we note that the continuity of the embedding $W^{1,4}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$ and estimate (18b) applied with fixed $p = 4$ imply that $h \in W^{1,\infty}(\mathbb{T}^2)$ and that there exists a constant c_1 independent of p such that

$$\|\boldsymbol{\nabla} h\|_{L^\infty} \leq \frac{c_1}{\sigma} \, \frac{1}{1-r} \,. \tag{31}$$

Then, by the Sobolev lemma, h is continuous. In the next proposition we derive sharp pointwise upper and lower bounds.

Proposition 4 ([5]). *Suppose $\tilde{q} \in L^\infty(\mathbb{T}^2)$ with $\|\tilde{q}\|_{L^\infty} < 1$ and let h be the solution to $L_q h = 1$ given by Proposition 3. Then*

$$\frac{1}{q_+} \leq h \leq \frac{1}{q_-} \,. \tag{32}$$

Proof. We rewrite the equation $L_q h = 1$ in the form

$$L_q \left(h - \frac{1}{q_+} \right) = 1 - \frac{q}{q_+} \geq 0 \,. \tag{33}$$

First, suppose that $q \in C(\mathbb{T}^2)$ and $h \in C^2(\mathbb{T}^2)$. Since L_q is uniformly elliptic, the classical strong maximum principle [7, 10] then implies

$$h - \frac{1}{q_+} \geq 0 \,. \tag{34}$$

The upper bound on h follows from the corresponding argument for $h - 1/q_-$. The general case when $\tilde{q} \in L^\infty(\mathbb{T}^2)$ follows by a standard mollification argument. □

We now proceed to studying weak solutions of $\Lambda_h \boldsymbol{u} = \boldsymbol{g}$. As usual, a weak solution is a vector field $\boldsymbol{u} \in H^1(\mathbb{T}^2, \mathbb{R}^2)$ which satisfies, for given $\boldsymbol{g} \in H^{-1}(\mathbb{T}^2, \mathbb{R}^2)$,

$$B(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle_{H^{-1}, H^1} \tag{35}$$

for every $\boldsymbol{v} \in H^1(\mathbb{T}^2, \mathbb{R}^2)$, where the the bilinear form B reads

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_{\mathbb{T}^2} \bigl(\boldsymbol{u} \cdot \boldsymbol{v} + \sigma \, h \, \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} - \sigma \, \boldsymbol{\nabla} h \cdot (\boldsymbol{\nabla} \boldsymbol{u})^T \boldsymbol{v} \bigr) \, \mathrm{d}\boldsymbol{x} \tag{36}$$

and the colon denotes summation of componentwise products over both indices. The properties of the full potential vorticity inversion are now stated in the following proposition.

Proposition 5. *Suppose $\tilde{q} \in L^\infty(\mathbb{T}^2)$ with $\|\tilde{q}\|_{L^\infty} \leq r < 1$. Let h denote the solution to $L_q h = 1$ given by Proposition 3. Further, for $2 \leq p < \infty$, let $\boldsymbol{g} \in W^{-1,p}(\mathbb{T}^2, \mathbb{R}^2)$. Then the problem*

$$\Lambda_h \boldsymbol{u} = \boldsymbol{g} \tag{37}$$

has a unique weak solution $\boldsymbol{u} \in W^{1,p}(\mathbb{T}^2, \mathbb{R}^2)$ and there exists a constant c independent of p such that

$$\|\boldsymbol{u}\|_{W^{1,p}} \leq \frac{c\,p}{\sigma^2} \, \frac{1}{1 - r} \, \|\boldsymbol{g}\|_{W^{-1,p}} \,. \tag{38}$$

In particular, when $\boldsymbol{g}$ denotes the right hand side of the generalized LSG momentum equation (2b)*, then there exists a constant C independent of p but dependent on all other parameters as well as on r such that the velocity field $\boldsymbol{u} \equiv \boldsymbol{K}(q)$ is bounded by*

$$\|\boldsymbol{u}\|_{W^{1,p}} \leq C \, p \,. \tag{39}$$

Furthermore, $\boldsymbol{K}$ is uniformly continuous on the set $\{q = 1 + \tilde{q} \colon \|\tilde{q}\|_{L^\infty} \leq r\}$ as a map from $W^{-1,p}(\mathbb{T}^2)$ into $L^p(\mathbb{T}^2, \mathbb{R}^2)$. Specifically, there exists a constant C depending on r, p and on all parameters such that

$$\|\boldsymbol{K}(q_1) - \boldsymbol{K}(q_2)\|_{L^p} \leq C \, \|q_1 - q_2\|_{W^{-1,p}} \tag{40}$$

so long as $\|\tilde{q}_i\|_{L^\infty} \leq r$ for $i = 1, 2$.

Proof. Estimates (38) and (39) were already used in [5], so we give only a brief sketch of the proof. First note that $\boldsymbol{g} \in H^{-1}(\mathbb{T}^2, \mathbb{R}^2)$. We establish existence of a unique weak solution $\boldsymbol{u} \in H^1(\mathbb{T}^2, \mathbb{R}^2)$ by the Lax–Milgram theorem. Continuity of the bilinear form (36) is immediate. To prove coercivity, we write

$$\begin{aligned} B(\boldsymbol{u}, \boldsymbol{u}) &= \int_{\mathbb{T}^2} \bigl(|\boldsymbol{u}|^2 + \sigma\, h\, |\boldsymbol{\nabla u}|^2 - \tfrac{1}{2}\, \sigma\, \boldsymbol{\nabla} h \cdot \boldsymbol{\nabla} |\boldsymbol{u}|^2\bigr)\, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{T}^2} \bigl(\tfrac{1}{2}\, (1 + qh)\, |\boldsymbol{u}|^2 + \sigma\, h\, |\boldsymbol{\nabla u}|^2\bigr)\, \mathrm{d}\boldsymbol{x} \\ &\geq \min\{\tfrac{1}{2}, \sigma h_-\}\, \|\boldsymbol{u}\|_{H^1} \,. \end{aligned} \tag{41}$$

Since, by Proposition 4, $h_- \geq 1/q_+ > 1/2$, and $\sigma \leq 1$ throughout, the Lax–Milgram theorem asserts existence of a unique weak solution $\boldsymbol{u} \in H^1$. To show that $\boldsymbol{u} \in W^{1,p}$, we note that $\Lambda_h \boldsymbol{u} = \boldsymbol{g}$ can be written

$$(1 - \sigma \Delta) \boldsymbol{u} = \frac{\boldsymbol{g}}{h} - \tilde{b}\, \boldsymbol{u} + \frac{2\sigma}{h}\, \boldsymbol{\nabla} h \cdot \boldsymbol{\nabla u} \,. \tag{42}$$

Estimate (38) follows from Lemma 2 by bounding each of the terms on the right in the $W^{-1,p}$ norm. Estimate (39) is obtained by inserting $\boldsymbol{g}$ as defined by the right hand side of (2b) into (38). The details can be found in [5].

To prove uniform continuity of $\boldsymbol{K}$, we first note that

$$L_q(h\boldsymbol{u}) = \Lambda_h \boldsymbol{u} \,. \tag{43}$$

For $i = 1, 2$, suppose $\|\tilde{q}_i\|_{L^\infty} \leq r$, set $h_i = L_{q_i}^{-1} 1$, $\boldsymbol{u}_i = \boldsymbol{K}(q_i)$, and write $\boldsymbol{g}_i$ to denote the respective right hand sides of the generalized LSG momentum equation (2b). Using (43), write

$$\begin{aligned} \boldsymbol{u}_1 - \boldsymbol{u}_2 &= \frac{1}{h_1 h_2} \bigl(h_2\, L_{q_1}^{-1} \boldsymbol{g}_1 - h_1\, L_{q_2}^{-1} \boldsymbol{g}_2\bigr) \\ &= \frac{h_2 - h_1}{h_1 h_2}\, L_{q_1}^{-1} \boldsymbol{g}_1 + \frac{(L_{q_1}^{-1} - L_{q_2}^{-1}) \boldsymbol{g}_1}{h_2} + \frac{L_{q_2}^{-1} (\boldsymbol{g}_1 - \boldsymbol{g}_2)}{h_2} \,. \end{aligned} \tag{44}$$

To prove (40), we take the L^p norm of all three terms on the right of (44). Note that it actually suffices to estimate the L^p norm of the numerators, because Proposition 4 provides uniform L^∞ bounds on h_1^{-1} and h_2^{-1}. Beginning with the first term on the right of (44), we apply the Hölder inequality, the Sobolev embedding theorem, and the uniform continuity estimate (19) with $f = 1$ to obtain

$$\begin{aligned} \|(h_2 - h_1)\, L_{q_1}^{-1} \boldsymbol{g}_1\|_{L^p} &\leq \|h_2 - h_1\|_{L^{2p}}\, \|L_{q_1}^{-1} \boldsymbol{g}_1\|_{L^{2p}} \\ &\leq c(p)\, \|h_2 - h_1\|_{W^{1,p}}\, \|L_{q_1}^{-1} \boldsymbol{g}_1\|_{L^{2p}} \\ &\leq C(p, \sigma)\, \|q_2 - q_1\|_{W^{-1,p}}\, \|L_{q_1}^{-1} \boldsymbol{g}_1\|_{L^{2p}} \,. \end{aligned} \tag{45}$$

Further, by Proposition 3,

$$\begin{aligned} \|L_{q_1}^{-1} \boldsymbol{g}_1\|_{L^{2p}} &\leq C_1\, \|\boldsymbol{g}_1\|_{W^{-1,2p}} \\ &\leq C_1\, \|h_1 - \mu\, (2\, h_1\, \Delta h_1 + |\boldsymbol{\nabla} h_1|^2)\|_{L^{2p}} \\ &\leq C_2\, \bigl(\|h_1\|_{L^\infty} + |\mu|\, (2\, \|h_1\|_{L^\infty}\, \|\Delta h_1\|_{L^\infty} + \|\boldsymbol{\nabla} h_1\|_{L^\infty}^2)\bigr) \,. \end{aligned} \tag{46}$$

All terms on the right are bounded by constants which depend only on the parameters, p and r due to the L^∞ bounds on q_1 and h_1, the identity $\Delta h_1 = (q_1 h_1 - 1)/\sigma$ implied by (2a), and estimate (31).

The second term on the right of (44) is readily estimated by applying (19) with $\boldsymbol{f} = \boldsymbol{g}_1$ and $s = 2p$. As shown in (46), $\boldsymbol{g}_1$ is uniformly bounded in $W^{-1,2p}$, so that

$$\|(L_{q_1}^{-1} - L_{q_2}^{-1})\boldsymbol{g}_1\|_{L^p} \leq C\,\|q_1 - q_2\|_{W^{-1,p}} \,. \tag{47}$$

To estimate the last term on the right of (44), we recall that $L_{q_2}^{-1}$ is bounded as a map from $W^{-2,p}$ into L^p, so that

$$\|L_{q_2}^{-1}(\boldsymbol{g}_1 - \boldsymbol{g}_2)\|_{L^p} \leq C(p,\sigma)\,\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{W^{-2,p}} \,. \tag{48}$$

To proceed, fix $\boldsymbol{v} \in W^{2,p'}$ and note that

$$\begin{aligned}\langle \boldsymbol{v}, \boldsymbol{g}_1 - \boldsymbol{g}_2\rangle_{W^{2,p'},W^{-2,p}} &= \langle \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, h_1 - h_2\rangle - 2\mu\,\langle \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, (h_1 - h_2)\,\Delta h_1\rangle \\ &\quad - 2\mu\,\langle \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, h_2\,\Delta(h_1 - h_2)\rangle - \mu\,\langle \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, \boldsymbol{\nabla}(h_1 - h_2) \cdot \boldsymbol{\nabla}(h_1 + h_2)\rangle \,. \end{aligned} \tag{49}$$

We estimate term-by-term. An upper bound on the second term implies one on the first, so we estimate, using the Hölder inequality,

$$\langle \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, (h_1 - h_2)\,\Delta h_1\rangle \leq \|\boldsymbol{v}\|_{W^{1,p'}}\,\|h_1 - h_2\|_{L^p}\,\|\Delta h_1\|_{L^\infty} \,. \tag{50}$$

Similarly,

$$\begin{aligned}\langle \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, h_2\,\Delta(h_1 - h_2)\rangle &= -\langle \boldsymbol{\nabla} h_2\,\boldsymbol{\nabla}^\perp \cdot \boldsymbol{v} + h_2\,\boldsymbol{\nabla}\boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, \boldsymbol{\nabla}(h_1 - h_2)\rangle \\ &\leq \big(\|\boldsymbol{\nabla} h_2\|_{L^\infty}\,\|\boldsymbol{v}\|_{W^{1,p'}} + \|h_2\|_{L^\infty}\|\boldsymbol{v}\|_{W^{2,p'}}\big)\,\|h_1 - h_2\|_{W^{1,p}} \end{aligned} \tag{51}$$

and

$$\langle \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}, \boldsymbol{\nabla}(h_1 - h_2) \cdot \boldsymbol{\nabla}(h_1 + h_2)\rangle \leq \|\boldsymbol{v}\|_{W^{1,p'}}\,\|h_1 - h_2\|_{W^{1,p}}\,\big(\|h_1\|_{W^{1,\infty}} + \|h_2\|_{W^{1,\infty}}\big) \,. \tag{52}$$

Recalling the uniform $W^{1,\infty}$ bounds on h_i and the uniform L^∞ bound on $\Delta h_1 = \frac{1}{\sigma}(q_1 h_1 - 1)$, we find that, altogether,

$$\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{W^{-2,p}} \leq C_1\,\|h_1 - h_2\|_{W^{1,p}} \leq C_2\,\|q_1 - q_2\|_{W^{-1,p}} \,, \tag{53}$$

where the final inequality is due to (19) with $f = 1$. This completes the proof. □

As the right hand side of the momentum equation (2b) is divergence free, we obtain the following estimates for the divergence of $\boldsymbol{u}$ which are almost as strong as those for $\boldsymbol{u}$.

Proposition 6. *Suppose $\tilde{q} \in L^\infty(\mathbb{T}^2)$ with $\|\tilde{q}\|_{L^\infty} \leq r < 1$. Let $\boldsymbol{u} \equiv \boldsymbol{K}(q)$ denote the solution to $\Lambda_h \boldsymbol{u} = \boldsymbol{g}$ given by Proposition 5 and let $p \in [2,\infty)$. Then $\boldsymbol{\nabla} \cdot \boldsymbol{u} \in W^{1,p}$ and there exists a constant C depending on r, p, and all parameters such that*

$$\|\boldsymbol{\nabla} \cdot \boldsymbol{u}\|_{W^{1,p}} \leq C \,. \tag{54}$$

Furthermore, for every $0 \leq r < \sqrt{5}-2$ the operator $\boldsymbol{\nabla} \cdot \boldsymbol{K}$ is uniformly continuous on the set $\{q = 1 + \tilde{q} \colon \|\tilde{q}\|_{L^\infty} \leq r\}$ as a map from $W^{-1,p}(\mathbb{T}^2)$ into $L^p(\mathbb{T}^2)$. Specifically, there exists a constant C depending on r, p and on all parameters such that

$$\|\boldsymbol{\nabla} \cdot (\boldsymbol{K}(q_1) - \boldsymbol{K}(q_2))\|_{L^p} \leq C\,\|q_1 - q_2\|_{W^{-1,p}} \tag{55}$$

so long as $\|\tilde{q}_i\|_{L^\infty} \leq r$ for $i = 1, 2$.

Remark 1. In our proof, the dependence of the constant in (54) is cubic in contrast to the linear p-dependence of the corresponding estimate for $\boldsymbol{K}$. While the latter is essential for proving uniqueness of the solution to the full time-dependent problem, the p-dependence in (54), luckily, plays no further role.

Proof. Taking the divergence of $\Lambda_h \boldsymbol{u} = \boldsymbol{g}$, noting that $\boldsymbol{\nabla} \cdot \boldsymbol{g} = 0$, we obtain

$$\begin{aligned} \Lambda_h(\boldsymbol{\nabla} \cdot \boldsymbol{u}) &= \sigma \left(\boldsymbol{\nabla} h \cdot \Delta \boldsymbol{u} + 2\, \boldsymbol{\nabla}\boldsymbol{\nabla} h : \boldsymbol{\nabla} \boldsymbol{u} \right) \\ &= \sigma \left(\boldsymbol{\nabla} \cdot ((\boldsymbol{\nabla} \boldsymbol{u})^T \boldsymbol{\nabla} h) + \boldsymbol{\nabla}\boldsymbol{\nabla} h : \boldsymbol{\nabla} \boldsymbol{u} \right) \equiv \tilde{g} \,. \end{aligned} \tag{56}$$

We estimate

$$\begin{aligned} \|\tilde{g}\|_{W^{-1,p}} &= \sigma \sup_{\substack{\phi \in W^{1,p'} \\ \phi \neq 0}} \frac{\langle \boldsymbol{\nabla} \phi, (\boldsymbol{\nabla} \boldsymbol{u})^T \boldsymbol{\nabla} h \rangle + \langle \phi, \boldsymbol{\nabla}\boldsymbol{\nabla} h : \boldsymbol{\nabla} \boldsymbol{u} \rangle}{\|\phi\|_{W^{1,p'}}} \\ &\leq \sigma \left(\|(\boldsymbol{\nabla} \boldsymbol{u})^T \boldsymbol{\nabla} h\|_{L^p} + \|\boldsymbol{\nabla}\boldsymbol{\nabla} h : \boldsymbol{\nabla} \boldsymbol{u}\|_{L^p} \right) \\ &\leq 2\, \sigma \, \|\boldsymbol{u}\|_{W^{1,2p}} \, \|h\|_{W^{2,2p}} \,. \end{aligned} \tag{57}$$

The right hand norms are bounded due to (39) and (18b). Hence, $\tilde{g} \in W^{-1,p}$ so that Proposition 5 implies $\boldsymbol{\nabla} \cdot \boldsymbol{u} \in W^{1,p}$ with upper bound

$$\|\boldsymbol{\nabla} \cdot \boldsymbol{u}\|_{W^{1,p}} \leq \frac{c\,p}{\sigma} \, \frac{1}{1-r} \, \|\tilde{g}\|_{W^{-1,p}} \leq \frac{1}{\sigma} \, \frac{C}{(1-r)^2} \, p^3 \,. \tag{58}$$

This establishes (54).

To prove uniform continuity of $\boldsymbol{\nabla} \cdot \boldsymbol{K}$, we follow the proof of Proposition 5. Observe that, as in (43),

$$L_q(h \boldsymbol{\nabla} \cdot \boldsymbol{u}) = \Lambda_h(\boldsymbol{\nabla} \cdot \boldsymbol{u}) \,, \tag{59}$$

so that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) = \frac{h_2 - h_1}{h_1 h_2} \, L_{q_1}^{-1} \tilde{g}_1 + \frac{(L_{q_1}^{-1} - L_{q_2}^{-1}) \tilde{g}_1}{h_2} + \frac{L_{q_2}^{-1}(\tilde{g}_1 - \tilde{g}_2)}{h_2} \,. \tag{60}$$

We take the L^p norm of each term on the right where, as before, it suffices to consider the numerators. Since

$$\|L_{q_1}^{-1} \tilde{g}_1\|_{L^{2p}} \leq c(p) \, \|L_{q_1}^{-1} \tilde{g}_1\|_{W^{1,p}} \leq C_1 \, \|\tilde{g}_1\|_{W^{-1,p}} \leq C_2 \,, \tag{61}$$

the first term on the right of (60) can be estimated as in (45). Similarly, due to the $W^{-1,2p}$ bound on $\tilde{g}_1$ implied by (54), the second term on the right of (60) can be estimated as in (47).

However, the third and last term on the right of (60) requires more thorough consideration. Adding and subtracting terms, we write

$$\tilde{g}_1 - \tilde{g}_2 = \sigma \, (\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5) \tag{62}$$

with

$$\Gamma_1 = \boldsymbol{\nabla} \cdot ((\boldsymbol{\nabla} \boldsymbol{u}_1)^T \boldsymbol{\nabla}(h_1 - h_2)) \,, \tag{63a}$$

$$\Gamma_2 = \boldsymbol{\nabla} \cdot \left((\boldsymbol{\nabla}(\boldsymbol{u}_1 - \boldsymbol{u}_2))^T \boldsymbol{\nabla} h_2 \right) , \tag{63b}$$

$$\Gamma_3 = \boldsymbol{\nabla}\boldsymbol{\nabla}(h_1 - h_2)) : \boldsymbol{\nabla} \boldsymbol{u}_1 \,, \tag{63c}$$

$$\Gamma_4 = \boldsymbol{\nabla}\boldsymbol{\nabla} h_2 : \boldsymbol{\nabla}(\boldsymbol{u}_1 - \boldsymbol{u}_2) - \Delta h_2 \, \boldsymbol{\nabla} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \,, \tag{63d}$$

$$\Gamma_5 = \Delta h_2 \, \boldsymbol{\nabla} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \,. \tag{63e}$$

When the operator $L_{q_2}^{-1}$ in the third term on the right of (60) acts on $\Gamma_1, \dots, \Gamma_4$, we consider it as a map from $W^{-2,p}$ to L^p. Hence, for these terms it suffices to derive fairly routine $W^{-2,p}$ bounds.

Beginning with Γ_1, we integrate by parts and estimate

$$\begin{aligned}\|\Gamma_1\|_{W^{-2,p}} &= \sup_{\substack{\phi\in W^{2,p'}\\ \phi\neq 0}} \frac{\langle \boldsymbol{\nabla}\phi, (\boldsymbol{\nabla}\boldsymbol{u}_1)^T\boldsymbol{\nabla}(h_1-h_2)\rangle}{\|\phi\|_{W^{2,p'}}} = \sup_{\substack{\phi\in W^{2,p'}\\ \phi\neq 0}} \frac{\langle \boldsymbol{\nabla}\boldsymbol{u}_1\boldsymbol{\nabla}\phi, \boldsymbol{\nabla}(h_1-h_2)\rangle}{\|\phi\|_{W^{2,p'}}} \\ &\le \sup_{\substack{\phi\in W^{2,p'}\\ \phi\neq 0}} \frac{\|\boldsymbol{u}_1\|_{W^{1,2p'}}\|\phi\|_{W^{1,2p'}}\|h_1-h_2\|_{W^{1,p}}}{\|\phi\|_{W^{2,p'}}} \le C\,\|q_1-q_2\|_{W^{-1,p}}\,.\end{aligned} \tag{64}$$

The first inequality above is due to a triple Hölder inequality. For the second inequality, we used (39) to estimate the norm of $\boldsymbol{u}_1$, the continuity of the embedding $W^{2,p'}\hookrightarrow W^{1,2p'}$, and the uniform continuity result (19).

Similarly, after two consecutive integrations by parts and application of Hölder's inequality, Γ_2 is estimated as

$$\begin{aligned}\|\Gamma_2\|_{W^{-2,p}} &= \sup_{\substack{\phi\in W^{2,p'}\\ \phi\neq 0}} \frac{\langle \Delta\phi\,\boldsymbol{\nabla}h_2 + \boldsymbol{\nabla}\boldsymbol{\nabla}h_2\boldsymbol{\nabla}\phi, \boldsymbol{u}_1-\boldsymbol{u}_2\rangle}{\|\phi\|_{W^{2,p'}}} \\ &\le \sup_{\substack{\phi\in W^{2,p'}\\ \phi\neq 0}} \frac{\|\phi\|_{W^{2,p'}}\|\boldsymbol{\nabla}h_2\|_{L^\infty} + \|h_2\|_{W^{2,2p'}}\|\phi\|_{W^{1,2p'}}}{\|\phi\|_{W^{2,p'}}}\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^p} \\ &\le C\,\|q_1-q_2\|_{W^{-1,p}}\,.\end{aligned} \tag{65}$$

Besides the Sobolev embedding as in (64), estimates (18b), (31), and (40) were used in the last step of (65).

To estimate Γ_3, take a test function $\phi\in W^{2,p'}$ and repeatedly integrate by parts:

$$\begin{aligned}\langle\phi,\Gamma_3\rangle &= \langle \boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\boldsymbol{u}_1\boldsymbol{\nabla}(h_1-h_2)\rangle \\ &\quad + \langle \boldsymbol{\nabla}\phi, \boldsymbol{\nabla}(h_1-h_2)\,\boldsymbol{\nabla}\cdot\boldsymbol{u}_1\rangle + \langle \phi, \Delta(h_1-h_2)\,\boldsymbol{\nabla}\cdot\boldsymbol{u}_1\rangle\,.\end{aligned} \tag{66}$$

The first two terms are bounded as in (64). For the last term, use

$$\Delta(h_1-h_2) = \frac{1}{\sigma}(q_1(h_1-h_2) - h_2(q_2-q_1)) \tag{67}$$

to replace the Laplacian of the difference with lower order terms. Then,

$$\begin{aligned}|\sigma\,\langle\phi,\Delta(h_1-h_2)\,\boldsymbol{\nabla}\cdot\boldsymbol{u}_1\rangle| &\le \|\phi\|_{L^{2p'}}\|q_1\|_{L^\infty}\|h_1-h_2\|_{L^p}\|\boldsymbol{u}_1\|_{W^{1,2p'}} \\ &\quad + \|h_2\,(q_2-q_1)\|_{W^{-1,p}}\|\phi\,\boldsymbol{\nabla}\cdot\boldsymbol{u}_1\|_{W^{1,p'}} \\ &\le C_1\left(\|\phi\|_{W^{1,p'}}\|\boldsymbol{u}_1\|_{W^{1,2p'}} + \|\phi\|_{W^{2,p'}}\|\boldsymbol{\nabla}\cdot\boldsymbol{u}_1\|_{W^{1,p'}}\right)\|q_1-q_2\|_{W^{-1,p}} \\ &\le C_2\,\|\phi\|_{W^{2,p'}}\|q_1-q_2\|_{W^{-1,p}}\,.\end{aligned} \tag{68}$$

In the second step, we applied the Sobolev embedding theorem, the L^∞ bound on q_1, (19), and (30) from the proof of Proposition 3; the last step uses (39) and (58).

For Γ_4, take once again $\phi\in W^{2,p'}$ and repeatedly integrate by parts, so that

$$\langle\phi,\Gamma_4\rangle = \langle\boldsymbol{\nabla}\phi,(\boldsymbol{u}_1-\boldsymbol{u}_2)\,\Delta h_2\rangle - \langle\boldsymbol{\nabla}\phi,\boldsymbol{\nabla}\boldsymbol{\nabla}h_2\,(\boldsymbol{u}_1-\boldsymbol{u}_2)\rangle\,. \tag{69}$$

Both expressions can be estimated analogous to the second summand in (65).

The remaining and most critical contribution to (60) involves Γ_5. In an inner product $\langle\phi,\Gamma_5\rangle$, we could not move derivatives onto the test function without creating terms with third derivatives on h_2 which we cannot control. We must therefore look at the remaining term including all prefactors and consider $L_{q_2}^{-1}$ as a map from L^p into itself. Using (18a) and the substitution $q_2 h_2 - 1 = \sigma \Delta h_2$, we estimate

$$\|\sigma h_2^{-1} L_{q_2}^{-1} \Gamma_5\|_{L^p} \leq \alpha \, \|\boldsymbol{\nabla} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^p} \tag{70}$$

where

$$\alpha = \frac{\|h_2^{-1}\|_{L^\infty}}{1 - \|\tilde{q}_2\|_{L^\infty}} \, \|q_2 h_2 - 1\|_{L^\infty} \, . \tag{71}$$

Observe that $\|\boldsymbol{\nabla} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^p}$ is exactly the quantity we are trying to control. Thus, we may subtract (70) from both sides of the L^p estimate of (60) provided the prefactor α is strictly less than one. To derive a sufficient condition under which this is true, we apply the maximum principle bounds from Proposition 4 to obtain

$$\|h_2^{-1}\|_{L^\infty} \, \|q_2 h_2 - 1\|_{L^\infty} \leq q_+ \, \max\{\tfrac{q_+}{q_-} - 1, -(\tfrac{q_-}{q_+} - 1)\} \leq q_+ \left(\tfrac{q_+}{q_-} - 1\right) . \tag{72}$$

Now, setting $x \equiv \|\tilde{q}\|_{L^\infty}$ such that $q_+ = 1 + x$ and $q_- = 1 - x$, we obtain

$$\alpha \leq \frac{1+x}{1-x} \left(\frac{1+x}{1-x} - 1 \right) = 2x \, \frac{1+x}{(1-x)^2} \tag{73}$$

which lies in the interval $[0, 1)$ provided $0 \leq x < \sqrt{5} - 2 \approx 0.236$. □

4. Global weak solutions

In this section, we prove Theorem 1 in four Steps. The structure of the argument is classical; our presentation closely follows [11].

Step 1. *Construct a family of approximate solutions $\{q_\nu\}$.*

For each $\nu > 0$, we take the regularized initial potential vorticity $q_\nu^{\mathrm{in}} \equiv q^{\mathrm{in}} * j_\nu$, where j_ν denotes a scaled standard mollifier. Hence, q_ν^{in} is smooth, in particular of class H^3, so that, by [5, Theorem 5.2], the gLSG equations possess a unique global classical solution

$$q_\nu \in C([0,\infty); H^3(\mathbb{T}^2)) \cap C^1([0,\infty); H^2(\mathbb{T}^2)) \tag{74a}$$

$$\boldsymbol{u}_\nu \in C([0,\infty); H^4(\mathbb{T}^2, \mathbb{R}^2)) \cap C^1([0,\infty); H^3(\mathbb{T}^2, \mathbb{R}^2)) \tag{74b}$$

with $q_\nu(0) = q_\nu^{\mathrm{in}}$ which preserves the PV maximum and minimum in time. Since convolution of an L^∞ function with a standard mollifier preserves essential supremum and infimum, we obtain the sequence of potential vorticity bounds

$$0 < q_-^{\mathrm{in}} \leq (q_\nu^{\mathrm{in}})_- \leq q_\nu(\boldsymbol{x}, t) \leq (q_\nu^{\mathrm{in}})_+ \leq q_+^{\mathrm{in}} < \infty \, . \tag{75}$$

By direct computation, a strong gLSG solution is also a weak solution, i.e.

$$\langle \psi, q_\nu(t_2) \rangle - \langle \psi, q_\nu(t_1) \rangle - \int_{t_1}^{t_2} \langle \boldsymbol{\nabla} \cdot (\psi \boldsymbol{u}_\nu), q_\nu \rangle \, \mathrm{d}t = 0 \, , \tag{76a}$$

$$\boldsymbol{u}_\nu = \boldsymbol{K}(q_\nu) \, , \tag{76b}$$

$$q_\nu(0) = q_\nu^{\mathrm{in}} \, . \tag{76c}$$

We will now pass to the limit in this weak form. To this end, we investigate the compactness properties of $\{q_\nu\}$.

Step 2. *Show that $\{q_\nu\}$ is a relatively compact set in $C([0,\infty); \mathrm{w}^*\text{-}L^\infty(\mathbb{T}^2))$, in $\mathrm{w}^*\text{-}L^\infty([0,\infty) \times \mathbb{T}^2)$, in $C([0,\infty); \mathrm{w}\text{-}L^2(\mathbb{T}^2))$, and in $L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\mathbb{T}^2))$.*

Proof of Step 2. According to the Arzela–Ascoli theorem, $\{q_\nu\}$ is a relatively compact set in the space $C([0,\infty); \text{w}^*\text{-}L^\infty(\mathbb{T}^2))$, where $\text{w}^*\text{-}L^\infty(\mathbb{T}^2)$ denotes $L^\infty(\mathbb{T}^2)$ endowed with the weak-$*$ topology, provided the following is true:

(i) $\{q_\nu(t)\}$ is a relatively compact set in $\text{w}^*\text{-}L^\infty(\mathbb{T}^2)$ for every $t \in [0,\infty)$;

(ii) $\{q_\nu\}$ is uniformly equicontinuous in $C([0,\infty); \text{w}^*\text{-}L^\infty(\mathbb{T}^2))$, i.e. for every $\psi \in L^1$ the sequence $\{\langle q_\nu(\cdot), \psi\rangle_{L^\infty, L^1}\}$ is uniformly equicontinuous in $C([0,\infty))$.

Condition (i) is equivalent to $\{q_\nu(t)\}$ being bounded in L^∞ for every $t \in [0,\infty)$, hence is a consequence of (75). This also implies, in passing, that $\{q_\nu\}$ is relatively compact in $\text{w}^*\text{-}L^\infty([0,\infty) \times \mathbb{T}^2)$.

To show condition (ii), we first assume that ψ is smooth, so that

$$\begin{aligned}
\big|\langle q_\nu(t_2), \psi\rangle_{L^\infty, L^1} - \langle q_\nu(t_1), \psi\rangle_{L^\infty, L^1}\big| &= \left| \int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}_\nu + \psi\, \boldsymbol{\nabla} \cdot \boldsymbol{u}_\nu, q_\nu \big\rangle \,\mathrm{d}t \right| \\
&\leq 2\, \|\psi\|_{W^{1,\infty}} \max_{t_1 \leq t \leq t_2} \|q_\nu(t)\|_{L^\infty} \left(\int_{t_1}^{t_2} \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|\boldsymbol{u}_\nu(t)\|_{H^1}^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq C(q^{\text{in}})\, \|\psi\|_{W^{1,\infty}}\, |t_2 - t_1| \,.
\end{aligned} \tag{77}$$

The last inequality is due to the L^∞ bounds (75) and the associated ν-independent $W^{1,p}$ bound on $\boldsymbol{u}_\nu$ given by Proposition 5. This proves equicontinuity for smooth ψ; the general case is argued by density as in [11]. We conclude that $\{q_\nu\}$ is relatively compact in $C([0,\infty); \text{w}^*\text{-}L^\infty(\mathbb{T}^2))$, hence also in $C([0,\infty); \text{w-}L^2(\mathbb{T}^2))$ due to the continuity of the embedding.

Finally, the relative compactness of $\{q_\nu\}$ in $L^2_{\text{loc}}([0,\infty); H^{-1}(\mathbb{T}^2))$ follows from the continuity of the embedding

$$C([0,\infty); \text{w-}L^2(\mathbb{T}^2)) \hookrightarrow L^2_{\text{loc}}([0,\infty); H^{-1}(\mathbb{T}^2)) \,. \tag{78}$$

A proof can be found, e.g., in [6]. □

Step 3. *Pass to the limit.*

Proof of Step 3. Step 2 asserts the convergence of a subsequence, for convenience still denoted $\{q_\nu\}$, to a limit

$$q \in C([0,\infty); \text{w}^*\text{-}L^\infty(\mathbb{T}^2)) \cap L^\infty([0,\infty) \times \mathbb{T}^2) \tag{79}$$

in the following sense:

$$q_\nu \to q \quad \text{in } C([0,\infty); \text{w}^*\text{-}L^\infty(\mathbb{T}^2)) \,, \tag{80a}$$

$$q_\nu \to q \quad \text{in } \text{w}^*\text{-}L^\infty([0,\infty) \times \mathbb{T}^2) \,, \tag{80b}$$

$$q_\nu \to q \quad \text{in } L^2_{\text{loc}}([0,\infty); H^{-1}(\mathbb{T}^2)) \,, \tag{80c}$$

$$q_\nu \to q \quad \text{in } \text{w-}L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^2)) \,. \tag{80d}$$

Convergences (80a), (80b), and (80c) follow directly from Step 2, whereas (80d) is strictly weaker than (80b). Furthermore, (80b) also implies

$$0 < q_-^{\text{in}} \leq q_-(t) \leq q(\boldsymbol{x}, t) \leq q_+(t) \leq q_+^{\text{in}} < \infty \,, \tag{81}$$

first for almost every $(\boldsymbol{x}, t) \in [0,\infty) \times \mathbb{T}^2$ and then for every $t \in [0,\infty)$ and a.e. $\boldsymbol{x} \in \mathbb{T}^2$ due to the continuity of q with respect to time.

The bounds in (81) show that $\boldsymbol{u}(t) = \boldsymbol{K}(q(t))$ is well-defined for all $t \in [0, \infty)$. From the strong convergence in (80c) and the uniform continuity of the operators $\boldsymbol{K}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{K}$ due to (40) and (55) we deduce that

$$\boldsymbol{u}_\nu \to \boldsymbol{u} \quad \text{in } L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^2, \mathbb{R}^2)), \tag{82a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_\nu \to \boldsymbol{\nabla} \cdot \boldsymbol{u} \quad \text{in } L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^2)). \tag{82b}$$

We proceed to show that q and $\boldsymbol{u}$ satisfy the weak vorticity equation (3). To this end, we will pass to the limit for each term in the weak formulation (76).

Let $\psi \in H^1(\mathbb{T}^2)$ be an arbitrary test function. Clearly, due to (80a),

$$\langle q_\nu(t), \psi \rangle \to \langle q(t), \psi \rangle \tag{83}$$

for every $t \in [0, \infty)$ as $\nu \to 0$, so that the first two terms in (76a) converge.

To show the convergence in the remaining term of (76a), we write

$$\int_{t_1}^{t_2} \langle \boldsymbol{\nabla} \cdot (\psi \boldsymbol{u}_\nu), q_\nu \rangle \, \mathrm{d}t = \int_{t_1}^{t_2} \langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}_\nu, q_\nu \rangle \, \mathrm{d}t + \int_{t_1}^{t_2} \langle \psi \, \boldsymbol{\nabla} \cdot \boldsymbol{u}_\nu, q_\nu \rangle \, \mathrm{d}t \,. \tag{84}$$

To pass to the limit in the first term on the right, we observe

$$\begin{aligned}\int_{t_1}^{t_2} \big(\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}_\nu, q_\nu \rangle - \langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}, q \rangle \big) \, \mathrm{d}t &= \int_{t_1}^{t_2} \big(\langle \boldsymbol{\nabla}\psi \cdot (\boldsymbol{u}_\nu - \boldsymbol{u}), q_\nu \rangle - \langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}, q - q_\nu \rangle \big) \, \mathrm{d}t \\ &\le \|\boldsymbol{\nabla}\psi\|_{L^2} \int_{t_1}^{t_2} \|\boldsymbol{u}_\nu - \boldsymbol{u}\|_{L^2} \, \|q_\nu\|_{L^\infty} \, \mathrm{d}t + \left| \int_{t_1}^{t_2} \langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}, q - q_\nu \rangle \, \mathrm{d}t \right| . \end{aligned} \tag{85}$$

The first term on the right of (85) vanishes as $\nu \to 0$ due to (75) and (82a). The second one vanishes due to the weak convergence (80d). Finally, due to (82b), the same argument applies to the second term on the right of (84). ☐

Step 4. *Show the uniqueness of the solution q and the continuous dependence of the solution on the initial data.*

Proof of Step 4. Let $\hat{q}$ and $\bar{q}$ be two weak solutions of problem (3) with common initial data $q^{\text{in}} \in L^\infty$ and write $\hat{\boldsymbol{u}} = \boldsymbol{K}(\hat{q})$, $\bar{\boldsymbol{u}} = \boldsymbol{K}(\bar{q})$. We shall establish the vanishing of $\varphi \equiv \hat{q} - \bar{q}$ in the H^{-1}-norm using

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\Phi(t)\|_{H^{-1}} = (\dot{\Phi}(t), \Phi(t))_{H^{-1}} \,, \tag{86}$$

which holds true in the sense of distributions on $\mathbb{R}$ for all $\Phi \in H^1_{\text{loc}}([0,\infty); H^{-1}(\mathbb{T}^2))$. Equation (86) is obvious for functions in $C_0^\infty([0,\infty); H^{-1}(\mathbb{T}^2))$ and generalizes by density. To apply this identity to weak gLSG solutions, we first note that $q \in L^2_{\text{loc}}([0,\infty); H^{-1}(\mathbb{T}^2))$ due to (80d). Further, we need to recover $\dot{q}$ in a weak sense. To this end, we fix a test function $\psi \in H^1$ and take $t_1 = 0$, $t_2 = t$ in the weak formulation (3a), so that

$$\langle \psi, q(t) \rangle = \langle \psi, q^{\text{in}} \rangle + \int_0^t \langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}(t'), q(t') \rangle \, \mathrm{d}t' + \int_0^t \langle \psi \, \boldsymbol{\nabla} \cdot \boldsymbol{u}(t'), q(t') \rangle \, \mathrm{d}t' \,. \tag{87}$$

Note that the integrals are differentiable for almost every $t \in [0, \infty)$ due to the local summability of their integrands. For the first integrand, this is implied by the uniform bound on $\boldsymbol{u}$ due to (39) since

$$\int_0^T |\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}(t), q(t) \rangle|^2 \, \mathrm{d}t \le \|\psi\|_{H^1}^2 \int_0^T \|\boldsymbol{u}(t)\|_{L^2}^2 \, \mathrm{d}t \, \|q\|^2_{L^\infty([0,\infty)\times\mathbb{T}^2)} \le \|\psi\|_{H^1}^2 \, C \, T \,. \tag{88}$$

Due to (54), the argument for the second integrand is similar.

Now, multiply (87) with $\mathrm{d}\theta/\mathrm{d}t$, where θ is a test function $\theta \in C_0^\infty([0,\infty))$, integrate from 0 to ∞, and then integrate by parts to obtain

$$\int_0^\infty \langle \psi, q(t) \rangle \frac{\mathrm{d}\theta}{\mathrm{d}t}(t)\,\mathrm{d}t = -\int_0^\infty \big(\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}(t), q(t) \rangle + \langle \psi\, \boldsymbol{\nabla} \cdot \boldsymbol{u}(t), q(t) \rangle \big)\, \theta(t)\,\mathrm{d}t\,. \quad (89)$$

By definition, as $\langle \psi, q(\cdot) \rangle \in L^1_{\mathrm{loc}}([0,\infty))$, this means

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi, q(t) \rangle &= \langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{u}(t), q(t) \rangle + \langle \psi\, \boldsymbol{\nabla} \cdot \boldsymbol{u}(t), q(t) \rangle \\ &= \langle \psi, \boldsymbol{\nabla} \cdot \boldsymbol{u}(t)\, q(t) - \boldsymbol{\nabla} \cdot (\boldsymbol{u}(t)\, q(t)) \rangle_{H^1,H^{-1}} \end{aligned} \quad (90)$$

in the sense of distributions, hence

$$\dot{q}(t) = \boldsymbol{\nabla} \cdot \boldsymbol{u}(t)\, q(t) - \boldsymbol{\nabla} \cdot (\boldsymbol{u}(t)\, q(t))\,, \quad (91)$$

and, due to (88), $\dot{q} \in L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\mathbb{T}^2))$.

To proceed, we write $A \equiv 1 - \Delta$. Using its spectral representation, we can define arbitrary powers of A and, in particular, endow $H^{-1}(\mathbb{T}^2)$ with scalar product $\langle A^{-1/2}\,\cdot\,, A^{-1/2}\,\cdot\, \rangle$. Moreover,

$$\langle A^{-1}(\boldsymbol{\nabla} \cdot v), \phi \rangle = -\langle v, \boldsymbol{\nabla} A^{-1} \phi \rangle\,. \quad (92)$$

Then, by (86) and (92),

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(t)\|_{H^{-1}} = -\langle A^{-1}(\boldsymbol{\nabla} \cdot (\hat{\boldsymbol{u}}\hat{q}) - \hat{q}\, \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}), \varphi \rangle + \langle A^{-1}(\boldsymbol{\nabla} \cdot (\bar{\boldsymbol{u}}\bar{q}) - \bar{q}\, \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}), \varphi \rangle \\ &= \langle \hat{\boldsymbol{u}}\hat{q}, \boldsymbol{\nabla} A^{-1}\varphi \rangle + \langle \hat{q}\, \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}, A^{-1}\varphi \rangle - \langle \bar{\boldsymbol{u}}\bar{q}, \boldsymbol{\nabla} A^{-1}\varphi \rangle - \langle \bar{q}\, \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}, A^{-1}\varphi \rangle \\ &= \langle \bar{q}\, \boldsymbol{\nabla} \cdot (\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}), A^{-1}\varphi \rangle + \langle \bar{q}(\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}), \boldsymbol{\nabla} A^{-1}\varphi \rangle + \langle \varphi\, \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}, A^{-1}\varphi \rangle + \langle \hat{\boldsymbol{u}}\varphi, \boldsymbol{\nabla} A^{-1}\varphi \rangle\,. \end{aligned} \quad (93)$$

The first term on the right of (93) is estimated using the global L^∞-bound on $\bar{q}$ and the uniform continuity of $\boldsymbol{\nabla} \cdot \boldsymbol{K}$ as stated in (55), so that

$$\langle \bar{q}\, \boldsymbol{\nabla} \cdot (\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}), A^{-1}\varphi \rangle \le \|\bar{q}\|_{L^\infty} \|\boldsymbol{\nabla} \cdot (\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}})\|_{L^2} \|\varphi\|_{H^{-1}} \le C\, \|\varphi\|^2_{H^{-1}}\,. \quad (94)$$

Similarly, the second term is estimated by using the uniform continuity of $\boldsymbol{K}$ as formulated in (40),

$$\langle \bar{q}(\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}), \boldsymbol{\nabla} A^{-1}\varphi \rangle \le \|\bar{q}\|_{L^\infty} \|\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{L^2} \|\varphi\|_{H^{-1}} \le C\, \|\varphi\|^2_{H^{-1}}\,. \quad (95)$$

Setting $\psi = A^{-1}\varphi$, the third term on the right of (93) is estimated

$$\begin{aligned} &\langle \varphi\, \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}, A^{-1}\varphi \rangle = \langle A\psi, \psi\, \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} \rangle \\ &\quad = \langle \psi, \psi\, \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} \rangle + \langle \boldsymbol{\nabla}\psi, \psi\, \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} \rangle + \langle \boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi\, \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} \rangle \\ &\quad \le \|\psi\|^2_{L^4} \|\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}\|_{L^2} + \|\boldsymbol{\nabla}\psi\|_{L^2} \|\psi\|_{L^4} \|\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}\|_{L^4} + \|\boldsymbol{\nabla}\psi\|^2_{L^2} \|\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}\|_{L^\infty} \\ &\quad \le c\, \|\psi\|^2_{H^1} \|\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}\|_{W^{1,4}} \le C\, \|A^{-1}\varphi\|^2_{H^1} \le C\, \|\varphi\|^2_{H^{-1}}\,. \end{aligned} \quad (96)$$

Here, we employed the continuous embedding $W^{1,4} \hookrightarrow L^\infty$ in the fourth and (54) in the second to last step.

Finally, the last term on the right of (93) is estimated using integration by parts, the Hölder inequality, and the $W^{1,p}$-bound on $\boldsymbol{u}$ given by (39), so that

$$\begin{aligned}\langle A\psi, \hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\psi\rangle &= \langle \hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\psi, \psi\rangle + \langle(\boldsymbol{\nabla}\hat{\boldsymbol{u}})^T\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi\rangle - \tfrac{1}{2}\langle\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}, |\boldsymbol{\nabla}\psi|^2\rangle \\ &\le 3\,\|\hat{\boldsymbol{u}}\|_{W^{1,p}}\,\|\psi\|^2_{W^{1,2q}} \le C\,p\,\|A^{-1}\varphi\|^2_{W^{1,2q}} \\ &\le C\,p\,\|A^{-1/2}\varphi\|^2_{L^{2q}} \le C\,p\,\|\varphi\|^{2-2/p}_{H^{-1}}\,, \end{aligned} \tag{97}$$

where $1/p + 1/q = 1$ with $p \in [2,\infty)$ and, in the last inequality we used the boundedness of $A^{-1/2}\varphi \in L^\infty([0,\infty)\times\mathbb{T}^2)$.

Altogether, after absorbing excess powers of $\|\varphi\|_{H^{-1}}$ into the constant and redefining p as $2p$, we have derived the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|_{H^{-1}} \le C\,p\,\|\varphi(t)\|^{1-1/p}_{H^{-1}}\,. \tag{98}$$

Upon integration, we obtain the upper bound

$$\|\varphi(t)\|_{H^{-1}} \le \bigl(C\,t\bigr)^p\,. \tag{99}$$

Letting $p \to \infty$, we conclude that $\|\varphi(t)\|_{H^{-1}}$ vanishes on the interval $[0,t_0]$ for $C\,t_0 < 1$. Repeated application of this argument proves uniqueness on the entire time axis. We remark that this strategy is due to Yudovich [18, 19].

Finally, to prove stability, we use the above estimates without assuming that the initial data corresponding to $\hat{q}$ and $\bar{q}$ are identical. The differential inequality (98) remains valid. Upon integration, for any $\varepsilon > 0$, we find that if we choose $\delta(\varepsilon) \equiv 3^{-p(\varepsilon)}$ with $p(\varepsilon) > \max\{2, \ln\varepsilon/\ln(2/3)\}$, then for all admissible $\hat{q}$ and $\bar{q}$ with $\|\varphi(0)\|_{H^{-1}} \le \delta$ one has $\|\varphi(t)\|_{H^{-1}} < \varepsilon$ for any $t \in [0, 1/(3\,C)]$. Tiling the interval $[0,T]$ into equidistant subintervals, we obtain stability up to any finite time $T > 0$. □

5. Point Vortices

A brief inspection of the proof of Proposition 5 shows that when $\mu = 0$ in the gLSG momentum equation (2b), then $\boldsymbol{u} \in W^{3,p}$ for any $p \in [2,\infty)$. In other words, the potential vorticity inversion gains three derivatives in Sobolev space. Thus, the situation might appear similar to the Euler-α equations which are known to also possess weak solutions with Radon-measured potential vorticities [14]. However, due to the nonlinearity of the gLSG vorticity inversion, this analogy breaks down entirely.

We shall briefly demonstrate that there are no physical solutions to the first stage of the vorticity inversion, equation (2a), in the special case of the strictly positive Radon measure $q = 1 + \delta$, where δ denotes the Dirac measure. For convenience we set $\sigma = 1$ and consider the problem on $\mathbb{R}^2$. In this case, (2a) reads

$$h + \delta h - \Delta h = 1\,. \tag{100}$$

For δh to make sense as a Radon measure, the solution h must at least be continuous in the origin. Moreover, on the punctured plane $\mathbb{R}^2 \setminus \{0\}$, h must satisfy the Helmholtz equation $(1-\Delta)h = 1$. Since (100) has radial symmetry, a unique solution must be radial. Writing $h(r) = 1 + \tilde{h}(r)$, the radial Helmholtz equation reads

$$r^2\,\tilde{h}'' + r\,\tilde{h}' - r^2\,\tilde{h} = 0\,. \tag{101}$$

Solutions are the modified Bessel functions I_0 and K_0 [1]. K_0 has a logarithmic singularity at 0 and cannot be extended to a continuous function on $\mathbb{R}^2$. I_0 is continuous at the origin, but grows exponentially when $r \to \infty$, hence must be considered unphysical. We conclude that not even a single potential vorticity point vortex is supported by our model.

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