A variational derivation of the geostrophic momentum approximation

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This paper demonstrates that the shallow water semigeostrophic equations arise from a degenerate second order Hamilton principle of very special structure. The associated Euler–Lagrange operator factors into a fast and a slow first order operator; restricting to the slow part yields the geostrophic momentum approximation as balanced dynamics. While semigeostrophic theory has been considered variationally before, this structure appears to be new. It leads to a straightforward derivation of the geostrophic momentum approximation and its associated potential vorticity law. Our observations further affirm, from a different point of view, the known difficulty in generalizing the semigeostrophic equations to the case of a spatially varying Coriolis parameter.

1. Introduction

A variational derivation of balance models for rotating shallow water in the semigeostrophic limit was first suggested by Salmon (1983, 1985, 1988), who used the leading order balance relation to constrain the Hamilton principle. By reversing the order of Salmon's two steps, namely the application of a constraint and a change of coordinates, the author was subsequently able to provide a more flexible variational construction which can be used, in principle, to derive balance models at any order of accuracy (Oliver, 2006). A particular advantage of this new construction is the ability to include the case of spatially varying Coriolis parameter in a straightforward manner (Oliver & Vasylkevych, 2013).

The classical treatment of this limit, which is also known as *Phillips type 2 scaling* (Phillips, 1963) or the *frontal dynamics regime* (Reznik *et al.*, 2001), goes back to Eliassen (1948, 1962), who introduced the *geostrophic momentum approximation*, where the advected velocity, but not the advecting velocity, is replaced by the geostrophic velocity. Hoskins (1975) realized that the resulting dynamics is described by potential vorticity advection in suitably changed coordinates. Horizontal plane versions of this transformation even go back to Yudin (1955) as remarked in Blumen (1981), and to Eliassen (1962). Now known as the Hoskins transformation, it has subsequently been interpreted as a Legendre duality (Cullen & Purser, 1984; Cullen & Purser, 1988; McIntyre & Roulstone, 2002), and can be constructed as the solution of an optimal transportation problem (Benamou & Brenier, 1998; Cullen, 2006). The connection with optimal transportation has raised considerable interest in semigeostrophic theory from the mathematical community, leading to a number of rigorous results on the solution theory of semigeostrophic equations in various settings (Cullen & Gangbo, 2001; Jian & Wang, 2007; Cullen, 2008; Ambrosio *et al.*, 2012; Figalli, 2013).

The purpose of this paper is to clarify the role of the shallow water semigeostrophic equations within the variational approach to balanced dynamics. In particular, we shall derive the geostrophic momentum approximation from a variational ansatz, which is a complete reversal of the classical argument. As an aside, we illustrate how the semigeostrophic potential vorticity law arises from the particle relabeling symmetry. We further identify the special structure of the semigeostrophic Lagrangian which is related to but structurally different from Salmon's L_1 Lagrangian and its subsequent generalizations. In particular, we show that the semigeostrophic Lagrangian is an affine second order Lagrangian. Further, the Hoskins transformation between physical and geostrophic coordinates can be identified with a self-adjoint differential operator in the time domain which commutes with time differentiation and with the symplectic structure matrix. While a variational approach to the shallow water semigeostrophic equations has been considered before (Salmon, 1988; Oliver, 2006), this semigeostrophic Hamilton principle has, to the best of our knowledge, not been spelled out explicitly before.

For large-scale motion in the atmosphere, as opposed to the ocean where the radius of deformation is much smaller and outside of the frontal scaling used by Hoskins, the geostrophic momentum approximation is only valid on scales where the variation of the Coriolis parameter f is not negligible. Thus, there is considerable interest in developing semigeostrophic theory which allows for spatially varying f. However, the structure of the geostrophic momentum approximation is very special so that the geometric structures in the case when f is a constant do not generalize in a canonical way. In our derivation, this can be traced to the emergence of commutator terms which are incompatible with the truncation to an affine second order Lagrangian. We consider it an open question whether there is a relaxed set of structure requirements which may lead to an equation of similarly appealing simplicity when f is spatially varying. This situation contrasts with the generalized large scale semigeostrophic equations, which generalize Salmon's L_1 dynamics and which do extend naturally to the case of non-constant f (Oliver & Vasylkevych, 2013). A different approach has been pursued by Cullen *et al.* (2005), who extended the optimal transport formulation of semigeostrophic motion to non-constant f without involving the equations of motion directly.

2. Setting

Our starting point is the rotating shallow water system which, in non-dimensionalized variables, reads

$$\varepsilon\,(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}) + f\,\boldsymbol{u}^\perp + \frac{\mathrm{Bu}}{\varepsilon}\,\boldsymbol{\nabla} h = 0\,, \tag{2.1a}$$

$$\partial_t h + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0 \tag{2.1b}$$

where $\boldsymbol{u} = (u_1, u_2)$ denotes the horizontal velocity, $\boldsymbol{u}^\perp = (-u_2, u_1)$, h denotes the layer depth, and f the non-dimensionalized Coriolis parameter, where we assume that either $f = 1$ or $f = O(1)$. The dynamical regime is characterized by the *Rossby number* $\varepsilon = U/(\Omega L)$ and the *Burger number* $\mathrm{Bu} = gH/(\Omega^2 L^2)$, where U, Ω, L, and H denote the characteristic velocity scale, rate of rotation, horizontal, and vertical length scales, respectively. Here, we consider the semigeostrophic regime where $\varepsilon = \mathrm{Bu} \ll 1$; we remark that quasi-geostrophic regime is a different distinguished limit where $\mathrm{Bu} = O(1)$ and $h = 1 + O(\varepsilon)$.

It is well known that the shallow water equations arise as the Euler–Lagrange equations from a Hamilton principle (e.g. Salmon, 1998). Our construction is based on this formulation.

In the following, we will always write $\boldsymbol{x}$ to denote an Eulerian position, $\boldsymbol{a}$ to denote a Lagrangian label, and $\boldsymbol{\eta}$ to denote the flow map, so that the particle initially at location

$\boldsymbol{a}$ has position $\boldsymbol{x} = \boldsymbol{\eta}(\boldsymbol{a}, t)$ at time t. All fields are functions of the Eulerian position; when we need to evaluate at the Lagrangian label, we write out the composition with the flow map explicitly. In particular, the relation between the Lagrangian and the Eulerian velocity is written as $\dot{\boldsymbol{\eta}}(\boldsymbol{a}, t) = \boldsymbol{u}(\boldsymbol{\eta}(\boldsymbol{a}, t), t)$, which we abbreviate as

$$\dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta} \,. \tag{2.2}$$

Similarly, the relation between the flow map and the Eulerian height field is given by $h(\boldsymbol{\eta}(\boldsymbol{a}, t), t) = h_0(\boldsymbol{a}) / \det \boldsymbol{\nabla}\boldsymbol{\eta}(\boldsymbol{a})$. Without loss of generality, we can set $h_0 = 1$ (then the initial flow map may deviate from the identity map), so that we can operate with the simplified short-hand expression

$$h \circ \boldsymbol{\eta} = \frac{1}{\det \boldsymbol{\nabla}\boldsymbol{\eta}} \,. \tag{2.3}$$

These conventions are common in the mathematical literature, less so in the geophysical fluid dynamics community. Nonetheless, we believe that there is clear advantage to our notation as it is unambiguous about implied dependences and, moreover, treats maps, vector fields, and coordinate labels as mathematically distinct objects.

By the Liouville theorem, equation (2.3) is equivalent to the continuity equation (2.1b). The momentum equation (2.1a) is the Euler–Lagrange equation for the shallow water Lagrangian in semigeostrophic scaling,

$$L = \int \bigl(\boldsymbol{R} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} + \tfrac{1}{2}\, \varepsilon\, |\dot{\boldsymbol{\eta}}|^2 - \tfrac{1}{2}\, h \circ \boldsymbol{\eta} \bigr) \, \mathrm{d}\boldsymbol{a} \,, \tag{2.4}$$

where $\boldsymbol{R}$ is a vector potential for the Coriolis parameter so that $\boldsymbol{\nabla}^\perp \cdot \boldsymbol{R} = f$. In other words, (2.1$a$) describes stationary points of the action

$$S = \int_{t_1}^{t_2} L \, \mathrm{d}t \tag{2.5}$$

with regards to variations $\delta\boldsymbol{\eta}$ of the flow map that vanish at the arbitrary end points t_1 and t_2. To see this, we compute the variation of each of the terms in turn. First,

$$\delta \int \boldsymbol{R} \circ \boldsymbol{\eta} \cdot \dot{\boldsymbol{\eta}} \, \mathrm{d}\boldsymbol{a} = \int \bigl((\boldsymbol{\nabla}\boldsymbol{R} \circ \boldsymbol{\eta}\, \delta\boldsymbol{\eta}) \cdot \dot{\boldsymbol{\eta}} + \boldsymbol{R} \circ \boldsymbol{\eta} \cdot \delta\dot{\boldsymbol{\eta}} \bigr) \, \mathrm{d}\boldsymbol{a} = - \int \dot{\boldsymbol{\eta}}^\perp \cdot \delta\boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} \,, \tag{2.6}$$

where we dropped a perfect time derivative in the last equality as it does not contribute to the Euler–Lagrange equation. Second,

$$\delta \int |\dot{\boldsymbol{\eta}}|^2 \, \mathrm{d}\boldsymbol{a} = -2 \int \ddot{\boldsymbol{\eta}} \cdot \delta\boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} \,, \tag{2.7}$$

again up to a perfect time derivative. Last, we compute

$$\begin{aligned} \delta \int h \circ \boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} &= \int \bigl(\delta h \circ \boldsymbol{\eta} + \boldsymbol{\nabla} h \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta} \bigr) \, \mathrm{d}\boldsymbol{a} \\ &= - \int h \, \boldsymbol{\nabla} \cdot (h\boldsymbol{w}) \, \mathrm{d}\boldsymbol{x} + \int \boldsymbol{\nabla} h \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} \\ &= \int h \, \boldsymbol{w} \cdot \boldsymbol{\nabla} h \, \mathrm{d}\boldsymbol{x} + \int \boldsymbol{\nabla} h \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} \\ &= 2 \int \boldsymbol{\nabla} h \circ \boldsymbol{\eta} \cdot \delta\boldsymbol{\eta} \, \mathrm{d}\boldsymbol{a} \,, \end{aligned} \tag{2.8}$$

where $\boldsymbol{w}$ is the Eulerian variation vector field implicitly defined via $\delta\boldsymbol{\eta} = \boldsymbol{w} \circ \boldsymbol{\eta}$ whence, by the Liouville theorem, δh must satisfy the "density Lin constraint" $\delta h + \boldsymbol{\nabla} \cdot (h\boldsymbol{w}) = 0$. Noting that the variation $\delta\boldsymbol{\eta}$ is arbitrary, (2.6–2.8) imply the momentum equation (2.1a).

3. Variational Asymptotics

The general construction follows Oliver (2006). We denote the flow map in the original physical coordinates by $\boldsymbol{\eta}_\varepsilon$ and take the point of view that ε parameterizes a family of near-identity deformations of the flow map which can be regarded as the flow of a change-of-variables vector field. The idea is motivated by generalized Lagrangian mean constructions (Andrews & McIntyre, 1978; Marsden & Shkoller, 2001; Holm, 2002; Salmon, 2013). The goal is to choose this change-of-variables order by order such that the variational principle and equations of motions take a convenient form.

Here, aiming at the semigeostrophic equations which are a second order model, we need to keep track of terms up to first order in ε. Hence, we make the ansatz

$$\boldsymbol{\eta}_\varepsilon = \boldsymbol{\eta} + \varepsilon\,\boldsymbol{\eta}' \tag{3.1}$$

where $\boldsymbol{\eta}$ will denote the flow map in the newly constructed frame and $\boldsymbol{\eta}'$, formally an expansion coefficient, will later be chosen as a function of $\boldsymbol{\eta}$ and its time derivatives.

We insert this ansatz into the shallow water Lagrangian and truncate terms at $O(\varepsilon^2)$ as convenient, so that

$$\begin{aligned} L_\varepsilon &= \int \big(\tfrac{1}{2}\,\boldsymbol{\eta}_\varepsilon^\perp \cdot \dot{\boldsymbol{\eta}}_\varepsilon + \tfrac{1}{2}\,\varepsilon\,|\dot{\boldsymbol{\eta}}_\varepsilon|^2 - \tfrac{1}{2}\,h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \big)\,\mathrm{d}\boldsymbol{a} \\ &= \int \big(\tfrac{1}{2}\,\boldsymbol{\eta}^\perp \cdot \dot{\boldsymbol{\eta}}_\varepsilon + \tfrac{1}{2}\,\varepsilon\,(\boldsymbol{\eta}'^\perp + \dot{\boldsymbol{\eta}}) \cdot \dot{\boldsymbol{\eta}}_\varepsilon - \tfrac{1}{2}\,h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \big)\,\mathrm{d}\boldsymbol{a} + O(\varepsilon^2)\,. \end{aligned} \tag{3.2}$$

In Oliver (2006), we characterized a class of variational balance model arising from affine first order Lagrangians. Here, we aim at obtaining an affine second order Lagrangian from (3.2). This is achieved by setting

$$\boldsymbol{\eta}' = \dot{\boldsymbol{\eta}}^\perp\,, \tag{3.3}$$

so that the semigeostrophic Lagrangian reads

$$L_{\mathrm{sg}} = \int \big(\tfrac{1}{2}\,\boldsymbol{\eta}^\perp \cdot \dot{\boldsymbol{\eta}}_\varepsilon - \tfrac{1}{2}\,h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \big)\,\mathrm{d}\boldsymbol{a}\,. \tag{3.4}$$

This Lagrangian is clearly affine. Moreover, due to the occurrence of a time derivative in the transformation (3.1), it is a Lagrangian of second order.

We now compute the Euler–Lagrange equation associated with L_{sg}. Later, in Section 5, we shall demonstrate that it implies the shallow water semigeostrophic equations in their usual form. Taking the variation of the semigeostrophic action and using (2.8) to compute the variation of the potential energy term, we obtain

$$\begin{aligned} \delta S_{\mathrm{sg}} &= \iint \big(\tfrac{1}{2}\,\delta\boldsymbol{\eta}^\perp \cdot \dot{\boldsymbol{\eta}}_\varepsilon + \tfrac{1}{2}\,\boldsymbol{\eta}^\perp \cdot \delta\dot{\boldsymbol{\eta}}_\varepsilon - \boldsymbol{\nabla} h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \cdot \delta\boldsymbol{\eta}_\varepsilon \big)\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t \\ &= \iint \big(\tfrac{1}{2}\,\delta\boldsymbol{\eta}^\perp \cdot \dot{\boldsymbol{\eta}} - \tfrac{1}{2}\,\varepsilon\,\delta\dot{\boldsymbol{\eta}}^\perp \cdot \boldsymbol{\eta}' - \tfrac{1}{2}\,\dot{\boldsymbol{\eta}}^\perp \cdot \delta\boldsymbol{\eta}_\varepsilon - \boldsymbol{\nabla} h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \cdot \delta\boldsymbol{\eta}_\varepsilon \big)\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t \\ &\quad + \int \big(\tfrac{1}{2}\,\varepsilon\,\delta\boldsymbol{\eta}^\perp \cdot \boldsymbol{\eta}' + \tfrac{1}{2}\,\boldsymbol{\eta}^\perp \cdot \delta\boldsymbol{\eta}_\varepsilon \big)\,\mathrm{d}\boldsymbol{a}\,\bigg|_{t_1}^{t_2} \\ &= \iint \delta\boldsymbol{\eta}_\varepsilon^\perp \cdot \big(\dot{\boldsymbol{\eta}} - \boldsymbol{\nabla}^\perp h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \big)\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t + \int \Big(\frac{1}{2}\,\boldsymbol{\eta}^\perp \cdot \delta\boldsymbol{\eta} + \frac{\varepsilon}{2}\,\frac{\partial}{\partial t}(\boldsymbol{\eta} \cdot \delta\boldsymbol{\eta}) \Big)\,\mathrm{d}\boldsymbol{a}\,\bigg|_{t_1}^{t_2}\,. \end{aligned} \tag{3.5}$$

Note that we have made use of (3.1) and (3.3) in several places, so that all steps in this computation are true identities to all orders. We have also kept all contributions from partial integration in time, as we will need to refer to these terms in Section 4 below.

Since L_{sg} is a second order Lagrangian, we impose that $\delta\boldsymbol{\eta}$ and $\delta\dot{\boldsymbol{\eta}}$ vanish at the temporal end points, so that the boundary terms in (3.5) drop out. When ε is small enough, the transformation (3.1) defines an invertible linear operator on the test function space, so that $\delta\boldsymbol{\eta}_\varepsilon$, for fixed label $\boldsymbol{a}$, can be chosen arbitrarily in the class of smooth and compactly supported test functions on $[t_1, t_2]$. These considerations show that the Euler–Lagrange equation reads

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\nabla}^\perp h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \tag{3.6}$$

or, setting $\boldsymbol{\eta}_\varepsilon = \boldsymbol{\xi}_\varepsilon \circ \boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta}$,

$$\boldsymbol{u} = \boldsymbol{\nabla}^\perp h_\varepsilon \circ \boldsymbol{\xi}_\varepsilon \,. \tag{3.7}$$

In other words, we recover the well-known relation that the velocity in geostrophic coordinates equals the geostrophic velocity in physical coordinates.

The argument above is formally correct, but it obscures that the configuration space of the balance model is the group of flow maps in *geostrophic* coordinates. This means that we really ought to write (3.5) entirely in terms of $\boldsymbol{\eta}$, using (3.1) and (3.3) to eliminate all references to $\boldsymbol{\eta}_\varepsilon$.

Let us therefore rethink the argument in these terms. To isolate the emerging structure from the details of the computation, we take a more abstract point of view, writing Φ to denote the transformation from geostrophic to physical coordinates. Then, $\boldsymbol{\eta}_\varepsilon = \Phi[\boldsymbol{\eta}]$, where $\Phi[\boldsymbol{\eta}] = \boldsymbol{\eta} + \varepsilon\, \dot{\boldsymbol{\eta}}^\perp$, so that Φ can be regarded as a first order differential operator on the time domain; we write Φ^* to denote its formal adjoint. Our particular transformation operator is evidently self-adjoint, but let us first proceed without assuming so.

We let J denote the canonical symplectic matrix, angle brackets denote the space-time L^2 inner product, and consider h as a functional acting on flow maps implicitly defined via (2.3). Then, in particular, $h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon \equiv h[\boldsymbol{\eta}_\varepsilon] = h[\Phi[\boldsymbol{\eta}]]$, so that

$$S_{\text{sg}} = \tfrac{1}{2} \left\langle \boldsymbol{\eta}, J \tfrac{\mathrm{d}}{\mathrm{d}t} \Phi[\boldsymbol{\eta}] \right\rangle + \tfrac{1}{2} \left\langle h[\Phi[\boldsymbol{\eta}]], 1 \right\rangle . \tag{3.8}$$

Referring to (2.8) as before, noting that differentiation and J commute with Φ, and dropping boundary terms from integration by parts, we obtain

$$\begin{aligned} \delta S_{\text{sg}} &= \tfrac{1}{2} \left\langle \delta\boldsymbol{\eta}, J\Phi[\dot{\boldsymbol{\eta}}] \right\rangle + \tfrac{1}{2} \left\langle \boldsymbol{\eta}, J\Phi[\delta\dot{\boldsymbol{\eta}}] \right\rangle + \left\langle \boldsymbol{u}_G[\Phi[\boldsymbol{\eta}]], \Phi[\delta\boldsymbol{\eta}] \right\rangle \\ &= \tfrac{1}{2} \left\langle \delta\boldsymbol{\eta}, \Phi[J\dot{\boldsymbol{\eta}}] \right\rangle + \tfrac{1}{2} \left\langle \delta\boldsymbol{\eta}, \Phi^*[J\dot{\boldsymbol{\eta}}] \right\rangle + \left\langle \delta\boldsymbol{\eta}, \Phi^*[\boldsymbol{u}_G[\Phi[\boldsymbol{\eta}]]] \right\rangle \end{aligned} \tag{3.9}$$

where the geostrophic velocity $\boldsymbol{u}_G$ is also regarded as a functional acting on flow maps defined via $\boldsymbol{u}_G[\boldsymbol{\eta}] = \boldsymbol{\nabla} h \circ \eta$.

The crucial point is that the Euler–Lagrange equation implied by (3.9) is second order in time. Thus, it is not a balance model in the usual sense. However, so long as Φ is self-adjoint, the Euler–Lagrange equation takes the special form

$$\Phi^*[J\dot{\boldsymbol{\eta}} + \boldsymbol{u}_G[\Phi[\boldsymbol{\eta}]]] = 0 \,, \tag{3.10}$$

i.e., it factors into an inner equation and an outer differential operator $\Phi^* = \Phi$ which is a singular perturbation of the identity. The inner equation

$$J\dot{\boldsymbol{\eta}} + \boldsymbol{u}_G[\Phi[\boldsymbol{\eta}]] = 0 \tag{3.11}$$

is the true balance model as it does not support motion on time scales faster than $O(\varepsilon^0)$; in concrete terms, it is given by (3.6) or (3.7). Any solution to (3.11) is clearly a solution to the full Euler–Lagrange equation (3.10). Conversely, however, (3.10) supports fast motion on $O(\varepsilon^{-1})$ time scales as well. Therefore, the variational procedure outlined above will only yield a balance model if the Euler–Lagrange equation has a fast-slow factorization as in (3.10). Direct calculation, however, shows that this structure is fragile: a varying Coriolis parameter will either lead to spatially varying J, which introduces unwanted commutator terms when taking the variational derivative, or destroy the self-adjointness of Φ, or both in an unstructured way.

4. Potential vorticity as Noetherian conservation law

We shall now give a direct derivation of the semigeostrophic potential vorticity equation as a Noetherian conservation law arising from the invariance of the semigeostrophic Lagrangian under particle relabeling. The connection between potential vorticity and particle relabeling is rather well known. Variants in the literature range from Salmon's (1998) textbook exposition, which uses entirely elementary component-based vector calculus, Bridges, Hydon & Reich's (2001) interpretation in terms of multisymplectic structure, and the entirely abstract Noether theorem in Marsden & Ratiu (1994) which can be shown to apply in this setting as well. The purpose of this section is to show that our somewhat non-standard semigeostrophic Lagrangian (3.4) falls into this setting and to give a complete and concise derivation of the potential vorticity law.

We begin by noting that a particle relabeling is a measure preserving transformation acting on labels $\boldsymbol{a}$. Let us consider a one-parameter family $\boldsymbol{\Phi}_\lambda$ of relabeling transformations chosen such that $\boldsymbol{\Phi}_0 = \mathrm{Id}$. The variation $\delta\boldsymbol{\Phi} = \partial\boldsymbol{\Phi}_\lambda/\partial\lambda\big|_{\lambda=0}$ then is a divergence free vector field which can be represented as the exterior derivative of a "variation stream function" θ, i.e., $\delta\boldsymbol{\Phi} = \boldsymbol{\nabla}^\perp\theta$.

Let us now take such particle relabeling variations about a trajectory $\boldsymbol{\eta}$ which satisfies the Euler–Lagrange equation (3.6). Then, $\boldsymbol{\eta}_\lambda(\boldsymbol{a},t) = \boldsymbol{\eta}(\boldsymbol{\Phi}_\lambda(\boldsymbol{a}),t)$, so that $\delta\boldsymbol{\eta} = \boldsymbol{\nabla}\boldsymbol{\eta}\,\delta\boldsymbol{\Phi} = \boldsymbol{\nabla}\boldsymbol{\eta}\,\boldsymbol{\nabla}^\perp\theta$. While such relabeling will never alter the Eulerian solution, the key observation is that the semigeostrophic Lagrangian remains invariant under particle relabeling as well, hence $\delta S_{\mathrm{sg}} = 0$. To proceed, we note that the computation performed in (3.5) holds true for variations about any one-parameter family of deformations of $\boldsymbol{\eta}$, so that we can reuse it here. As we now vary about a solution of the Euler–Lagrange equation, the double integral on the right of (3.5) vanishes and we are left with only the temporal boundary term contributions. I.e.,

$$\begin{aligned} 0 = \delta S &= \frac{1}{2}\int \boldsymbol{\eta}^\perp \cdot \delta\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{a}\bigg|_{t_1}^{t_2} + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{\eta}\cdot\delta\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{a}\bigg|_{t_1}^{t_2} \\ &= \frac{1}{2}\int \boldsymbol{\eta}^\perp \cdot \boldsymbol{\nabla}\boldsymbol{\eta}\,\boldsymbol{\nabla}^\perp\theta\,\mathrm{d}\boldsymbol{a}\bigg|_{t_1}^{t_2} + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{\eta}\cdot\boldsymbol{\nabla}\boldsymbol{\eta}\,\boldsymbol{\nabla}^\perp\theta\,\mathrm{d}\boldsymbol{a}\bigg|_{t_1}^{t_2} \\ &= -\int \det\boldsymbol{\nabla}\boldsymbol{\eta}\,\theta\,\mathrm{d}\boldsymbol{a}\bigg|_{t_1}^{t_2}, \end{aligned} \tag{4.1}$$

where we have integrated by parts in the last step and noted that the $O(\varepsilon)$ term drops out by anti-symmetry. As θ is arbitrary, this expression implies material conservation in geostrophic coordinates of the *potential vorticity*

$$q = \det\boldsymbol{\nabla}\boldsymbol{\eta}\circ\boldsymbol{\eta}^{-1} = \frac{1}{h}\,. \tag{4.2}$$

We first note that this is indeed a potential vorticity, as its dimensional form reads $q = f/h$. Second, following our notational convention, h is the height field in geostrophic coordinates, defined as the inverse Jacobian of the geostrophic coordinate flow map. Thus, equation (4.2) coincides with the familiar potential vorticity formula which is stated, for example, as equation (9.3) in McIntyre & Roulstone (2002), where the symbol h in their paper refers to the height field in physical coordinates, which is h_ε in our notation. This explains the appearance of the Jacobian of the Hoskins transformation in their formula.

5. Closing the evolution equation

Combining (3.1), (3.3), and the definition $\boldsymbol{\eta}_\varepsilon = \boldsymbol{\xi}_\varepsilon \circ \boldsymbol{\eta}$, we find that

$$\boldsymbol{\xi}_\varepsilon = \mathrm{id} + \varepsilon\, \boldsymbol{u}^\perp . \tag{5.1}$$

This expression is nothing but the well known *Hoskins transformation* between physical and semigeostrophic coordinates. Then, using the Eulerian form of the semigeostrophic Euler–Lagrange equation (3.7), we compute

$$\boldsymbol{\nabla}(h_\varepsilon \circ \boldsymbol{\xi}_\varepsilon) = \boldsymbol{\nabla} h_\varepsilon \circ \boldsymbol{\xi}_\varepsilon\, \boldsymbol{\nabla}\boldsymbol{\xi}_\varepsilon = -\boldsymbol{u}^\perp\,(I + \varepsilon\, \boldsymbol{\nabla}\boldsymbol{u}^\perp) = -\boldsymbol{u}^\perp - \tfrac{1}{2}\,\varepsilon\, \boldsymbol{\nabla}|\boldsymbol{u}|^2 . \tag{5.2}$$

This identity is special as it shows that $\boldsymbol{u}^\perp$ must be the gradient of a stream function. Writing $\boldsymbol{u} = \boldsymbol{\nabla}^\perp\psi$, recalling the definition $(\det \boldsymbol{\nabla}\boldsymbol{\eta}_\varepsilon)^{-1} = h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon$, and noting that the Jacobian of a composition is the product of the Jacobians, we have $h = h_\varepsilon \circ \boldsymbol{\xi}_\varepsilon \, \det \boldsymbol{\nabla}\boldsymbol{\xi}_\varepsilon$, which, by (5.2), is nothing but the Monge–Ampère equation

$$h = (\psi - \tfrac{1}{2}\,\varepsilon\,|\boldsymbol{\nabla}\psi|^2)\,\det(I - \varepsilon\,\boldsymbol{\nabla}\boldsymbol{\nabla}\psi) . \tag{5.3}$$

Advection of the potential vorticity implies that $h = 1/q$ is also advected in semi-geostrophic coordinates, i.e.,

$$\partial_t h + \boldsymbol{\nabla}^\perp\psi \cdot \boldsymbol{\nabla} h = 0 . \tag{5.4}$$

Equations (5.3) and (5.4) constitute a closed formulation for the shallow water semi-geostrophic equations in geostrophic coordinates. While this is well known (see, e.g., McIntyre & Roulstone 2002), equations (5.3) and (5.4) are usually derived from the Eulerian equations of motion by applying the geostrophic momentum approximation.

Here, we recover the geostrophic momentum approximation *a posteriori*. Namely, writing out the Hoskins transformation in Lagrangian variables,

$$\boldsymbol{\eta} = \boldsymbol{\eta}_\varepsilon + \varepsilon\, \boldsymbol{\nabla} h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon , \tag{5.5}$$

and differentiating in time, we find

$$\dot{\boldsymbol{\eta}} = \boldsymbol{u}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon + \varepsilon(\boldsymbol{\nabla}\dot{h}_\varepsilon \circ \boldsymbol{\eta}_\varepsilon + \boldsymbol{\nabla}\boldsymbol{\nabla} h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon\, \dot{\boldsymbol{\eta}}_\varepsilon) \equiv \boldsymbol{\nabla}^\perp h_\varepsilon \circ \boldsymbol{\eta}_\varepsilon . \tag{5.6}$$

This expression is clearly equivalent to the shallow water momentum equation in their geostrophic momentum approximation, which, setting $u_{\mathrm{G}} = \boldsymbol{\nabla}^\perp h_\varepsilon$, reads

$$\varepsilon\,(\partial_t + \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{u}_\varepsilon^\perp + \boldsymbol{\nabla} h_\varepsilon = 0 . \tag{5.7}$$

6. Discussion

Our derivation of the semigeostrophic equations from the shallow water Lagrangian is a reversal of Salmon's method: we first introduce a change of coordinates followed by a truncation of higher-order terms. As the resulting variational principle is degenerate, a balance constraint emerges by general theory in the form of a Dirac constraint; see Salmon (1988) for a discussion of the Dirac theory in the context of balance dynamics. This constraint is not needed in any explicit way. In this sense, our approach is different from that of Cullen *et al.* (1987) who derive semigeostrophic equations of motion via an energy minimization principle followed by an *explicitly* imposed constraint.

More concretely, we have shown that the semigeostrophic equations arise from a Hamilton principle with an affine second order Lagrangian L_{sg}. Its structure is such that one time derivative in its Euler–Lagrange equation can be factored out, so that the slow subsystem behaves like a balance model analogous to the balance models that arise generically from affine first order Lagrangians.

This point of view provides a structural justification for the geostrophic momentum approximation, usually justified purely in terms of asymptotic consistency. It also provides a different view on the problem of generalizing the semigeostrophic equations to a spatially varying Coriolis parameter. The question then can be phrased as follows. Can we generalize (3.1) in such a way that the computation in (3.5) still carries through such that the resulting Euler–Lagrange equation still factors? Direct computation, however, indicates that the commutator terms arising from non-constant f are a fundamental obstacle to obtaining a balance model in this framework. This situation is unlike that for the generalized large-scale semigeostrophic equations which extend naturally to the case of varying f, see Oliver & Vasylkevych (2013). At this time, we cannot strictly exclude the possibility that some relaxation of requirements could lead to a variable f semigeostrophic theory in this variational setting, but any such derivation would require considerable structural coincidences which derive from the abstract setting laid out at the end of Section 3. As pointed out by one of the referees, the work of Bridges, Hydon & Reich (2001) implies a similar geometric no-go result, although this is not explicitly stated in their paper.

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