ON THE EXISTENCE OF SOLUTIONS TO A BI-PLANAR MONGE–AMPÈRE EQUATION

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ABSTRACT. In this paper, we consider a fully nonlinear partial differential equation which can be expressed as a sum of two Monge–Ampère operators acting in different two-dimensional coordinate sections. This equation is elliptic, for example, in the class of convex functions. We show that the notion of Monge–Ampère measures and Aleksandrov generalized solutions extends to this equation, subject to a weaker notion of convexity which we call bi-planar convexity. While the equation is also elliptic in the class of bi-planar convex functions, the contrary is not necessarily true. This is a substantial difference compared to the classical Monge–Ampère equation where ellipticity and convexity coincide. We provide explicit counter-examples: classical solutions to the bi-planar equation that satisfy the ellipticity condition but are not generalized solutions in the sense introduced. We conclude that the concept of generalized solutions based on convexity arguments is not a natural setting for the bi-planar equation.

1. INTRODUCTION

We study the fully nonlinear second order equation

$$\partial_z^2\phi\,\partial_{x_1}^2\phi - (\partial_{x_1z}^2\phi)^2 + \partial_z^2\phi\,\partial_{x_2}^2\phi - (\partial_{x_2z}\phi)^2 = q \tag{1}$$

on a three-dimensional domain $\Omega \subset \mathbb{R}^3$. Setting

$$D_{x_jz}^2\phi \equiv \begin{pmatrix} \partial_{x_j}^2\phi & \partial_{x_jz}^2\phi \\ \partial_{zx_j}^2\phi & \partial_z^2\phi \end{pmatrix} \tag{2}$$

for $j = 1, 2$, we can write (1) in the form

$$\det D_{x_1z}^2\phi + \det D_{x_2z}^2\phi = q\,. \tag{3}$$

Thus, the operator on the left is the sum of two planar Monge–Ampère operators on perpendicular sections. For this reason, we shall refer to (3) as the *bi-planar Monge–Ampère equation*.

The characteristic matrix (see [3]) for (3) reads

$$\Lambda = \begin{pmatrix} \partial_{zz}\phi & 0 & -\partial_{x_1z}\phi \\ 0 & \partial_{zz}\phi & -\partial_{x_2z}\phi \\ -\partial_{x_1z}\phi & -\partial_{x_2z}\phi & \Delta\phi \end{pmatrix}, \tag{4}$$

where Δ denotes the Laplacian in the (x_1, x_2)-plane. Equation (3) is elliptic (in the sense of linearization) when Λ is positive definite. This is the case if and only if

$$\partial_{zz}\phi > 0 \quad \text{and} \quad \det D_{x_1z}^2\phi + \det D_{x_2z}^2\phi = q > 0\,. \tag{5}$$

In particular, (3) is elliptic in the class of convex functions.

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The study of the bi-planar Monge–Ampère equation is motivated by a recent paper [8] on variational balance models for rapidly rotating stratified fluid flow. For a class of models that includes the so-called L_1-model first proposed by R. Salmon [2, 10], the vertically integrated potential temperature Θ is related to the potential vorticity of the fluid via

$$\partial_z^2\Theta\,(1+\varepsilon\,\omega+\varepsilon\,\Delta\Theta)-\varepsilon\,(\partial_{zx_1}\Theta)^2-\varepsilon\,(\partial_{zx_2}\Theta)^2=q\,, \tag{6}$$

where, up to rescaling, $\omega=\omega(x_1,x_2)$ is the vorticity of the horizontal mean flow and ε is the Rossby number. Setting

$$\phi=\frac{1}{\sqrt{\varepsilon}}\,\Delta^{-1}(1+\varepsilon\,\omega)+\sqrt{\varepsilon}\,\Theta\,, \tag{7}$$

where Δ^{-1} denotes the inverse Laplacian on the two-dimensional horizontal domain U with homogeneous Dirichlet boundary conditions on ∂U, we see that (6) can be written in the form of the bi-planar Monge–Ampère equation (3) on the cylindrical domain $\Omega=U\times(0,1)$. In particular, (6) is elliptic if and only if $\partial_{zz}\Theta>0$ and $q>0$.

In this paper, we ask the question in which sense the well-established theory of generalized solution of the classical Monge–Ampère equation [3, 4, 5] carries over to the bi-planar Monge–Ampère equation. We find that it is possible to construct a bi-planar analog of the Monge–Ampère measure which can be used to define generalized solutions and assert their uniqueness [4, 5, 11]. For this construction, it is necessary to require that the solution is convex on the respective coordinate sections. This notion, which we term *bi-planar convexity*, is more general than convexity. However, it is also more restrictive than the ellipticity condition (5). Indeed, we show by example that there exist classical solutions to the Dirichlet problem for the bi-planar Monge–Ampère equation such that the equation is elliptic in the vicinity of these solutions; yet, these solutions are not bi-planar convex. This is in contrast to the situation for the classical Monge–Ampère equation equation where the notions of convexity and ellipticity coincide.

It should be noted that the classical Monge–Ampère equation equation is closely related to the geometric notion of convexity. However, bi-planar Monge–Ampère equation is related to the property of convexity for the two planar sections. Surely, if a function is convex, so are its planar sections. The converse however, is not true (see Remark 2 below). This illustrates that bi-planar convexity does not have an intrinsic geometric meaning in three dimensions. Correspondingly, convex analysis does not lead to a natural notion of solution for the bi-planar Monge–Ampère equation.

The remainder of the paper is structured as follows. In Section 2, we develop the convexity-based theory: we introduce the notion of bi-planar convexity, define the bi-planar Monge–Ampère measure, prove monotonicity and a comparison principle, and finally use these notions to define the bi-planar analog of Aleksandrov generalized solutions. Section 3 is devoted to counter-examples which show that there is a gap between the concept of convexity, or even bi-planar convexity, and ellipticity for associated Dirichlet problem. The paper concludes with a brief discussion.

2. Bi-planar Monge–Ampère measure

2.1. **Construction.** We define a measure on the Borel σ-algebra of $\mathbb{R}^3$, the bi-planar Monge–Ampère measure, by using planar Monge–Ampère measures on sections, then integrating over the remaining dimension. We begin the construction by defining a weaker notion of convexity adapted to the bi-planar structure of our equation.

Definition 1. Let ϕ be a continuous function defined on the set $\Omega \subset \mathbb{R}^3$. The function ϕ is *bi-planar convex* if for any fixed x_1 and x_2, $\phi_{x_1}(x_2, z) \equiv \phi(x_1, x_2, z)$ and $\phi_{x_2}(x_1, z) \equiv \phi(x_1, x_2, z)$ are convex functions on the respective sections

$$\Omega^{x_1} \equiv \{(x_2, z) \in \mathbb{R}^2 \colon (x_1, x_2, z) \in \Omega\} \tag{8}$$

and

$$\Omega^{x_2} \equiv \{(x_1, z) \in \mathbb{R}^2 \colon (x_1, x_2, z) \in \Omega\}\,, \tag{9}$$

whenever these are nonempty.

Remark 2. A convex function is bi-planar convex, but the converse is not necessarily true. For example,

$$\phi(x_1, x_2, z) = x_1^2 + x_2^2 + z^2 - 4\, x_1\, x_2 \tag{10}$$

is bi-planar convex but not convex.

For the classical Monge–Ampère equation on a domain $\Omega \subset \mathbb{R}^n$,

$$\det D^2\phi = \nu\,, \tag{11}$$

where ν is a given Borel measure on Ω, an Aleksandrov generalized solution is a convex function $\phi \in C(\Omega)$ such that $M\phi = \nu$, where $M\phi$ denotes the Monge–Ampère measure

$$M\phi(E) = |\partial\phi(E)| \tag{12}$$

for every Borel set $E \subset U$. Here $\partial\phi$ is the *normal map* or *subdifferential* defined at a point $x \in \Omega$ by

$$\partial\phi(x) = \{p \in \mathbb{R}^n \colon \phi(y) \geq \phi(x) + p \cdot (y - x) \text{ for all } y \in \Omega\} \tag{13}$$

and for a Borel set $E \subset \Omega$ by

$$\partial\phi(E) = \bigcup_{x \in E} \partial\phi(x)\,. \tag{14}$$

The Monge–Ampère measure (12) relates to the Monge–Ampère equation (11) via the identity

$$M\phi(E) = \int_E \det D^2\phi(x)\, dx \tag{15}$$

for all Borel sets $E \subset \Omega$, which holds true whenever $\phi \in C^2(\Omega)$; see, e.g., [4] for details. Derivatives of generalized solutions exist generally only in the sense of subdifferentials but, being convex, generalized solutions have classical derivatives of second order a.e. [1].

In the following, we mimic this correspondence for the bi-planar Monge–Ampère equation. Suppose $\Omega \subset \mathbb{R}^3$ is open and $\phi \in C(\overline{\Omega})$ is bi-planar convex. For every $x_1 \in \mathbb{R}$, we define the measure $M_{23}\phi_{x_1}$ on Ω^{x_1} as the planar Monge–Ampère measure associated with the convex continuous function ϕ_{x_1}; when Ω^{x_1} is empty, we take this measure to be zero. Likewise, for every $x_2 \in \mathbb{R}$, we define the measure $M_{13}\phi_{x_2}$ on Ω^{x_2} as the planar Monge–Ampère measure associated with ϕ_{x_2}.

When $E = E_1 \times E_{23} \subset \Omega$ is compact with $E_1 \subset \mathbb{R}$ and $E_{23} \subset \mathbb{R}^2$, then $\mu_{x_1}(E_{23}) \equiv M_{23}\phi_{x_1}(E_{23})$ is continuous on E_1 by [4, Lemma 1.2.3]. We can thus Lebesgue-integrate over E_1 and define

$$\mu_1(E) \equiv \int_{E_1} \mu_{x_1}(E_{23})\, dx_1 \,. \tag{16}$$

When $E = E_1 \times E_{23} \subset \Omega$ is open, we approximate E_{23} by an increasing sequence $K_1 \subset K_2 \subset K_3 \subset \dots$ of compact subsets such that $E_{23} = \cup_{n=1}^{\infty} K_n$. Correspondingly, the sequence of positive real numbers

$$\mu_1(E_1 \times K_n) \equiv \int_{E_1} \mu_{x_1}(K_n)\, dx_1 \tag{17}$$

is increasing, and

$$\mu_{x_1}(E_{23}) \equiv \lim_{n\to\infty} \mu_{x_1}(K_n) \tag{18}$$

is Lebesgue measurable on E_1. Hence, by the monotone convergence theorem,

$$\mu_1(E_1 \times E_{23}) \equiv \int_{E_1} \mu_{x_1}(E_{23})\, dx_1 = \lim_{n\to\infty} \int_{E_1} \mu_{x_1}(K_n)\, dx_1 \tag{19}$$

is well-defined. Due to the countable additivity of the Monge–Ampère measure and countable additivity of the Lebesgue integral, μ_1 defines a pre-measure which can be extended to a Borel measure on Ω.

Analogously, we define a measure μ_2 corresponding to the function ϕ_{x_2}. Finally, the bi-planar Monge–Ampère measure is defined as

$$\mu_\phi \equiv \mu_1 + \mu_2 \,. \tag{20}$$

2.2. **Basic properties.** We now provide the basic characterization of the bi-planar measure associated with smooth functions and prove monotonicity of the measure.

Lemma 3. *If $\phi \in C^2(\overline{\Omega})$ is bi-planar convex, then μ_ϕ is absolutely continuous with respect to the Lebesgue measure and*

$$\mu_\phi(E) = \int_E (\det D^2_{x_1 z}\phi + \det D^2_{x_2 z}\phi)\, dx_1\, dx_2\, dz \tag{21}$$

for all compact $E \subset \Omega$.

Proof. We will show that

$$\mu_1(E) = \int_E \det D^2_{x_2 z}\phi\, dx_1\, dx_2\, dz \,. \tag{22}$$

Due to the properties of the Lebesgue measure, it suffices to prove this relation for cylindrical sets of the form $E = E_1 \times E_{23} \subset \Omega$; an arbitrary compact $E \subset \Omega$ can be approximated by a union of such sets.

By [4, Example 1.1.14], we have

$$\mu_{x_1}(E_{23}) = \int_{E_{23}} \det D^2_{x_2 z}\phi\, dx_2\, dz \,. \tag{23}$$

Hence, by our definition (19), we have

$$\mu_1(E) = \int_{E_1} \mu_{x_1}(E_{23})\, dx_1 = \int_E \det D^2_{x_2 z}\phi\, dx_1\, dx_2\, dz \,. \tag{24}$$

Similar arguments show that

$$\mu_2(E) = \int_E \det D^2_{x_1 z}\phi \, dx_1 \, dx_2 \, dz \,. \tag{25}$$

Combining (22) and (25), we complete the proof. □

Lemma 4. *Let $\Omega \subset \mathbb{R}^3$ be open and bounded, and let $\phi, \psi \in C(\overline{\Omega})$ be bi-planar convex. If $\phi = \psi$ on $\partial\Omega$ and $\psi \geq \phi$ in Ω, then for any fixed $y_1 \in \mathbb{R}$ and $y_2 \in \mathbb{R}$,*

$$\partial\psi_{y_1}(\Omega^{y_1}) \subset \partial\phi_{y_1}(\Omega^{y_1}) \tag{26a}$$

and

$$\partial\psi_{y_2}(\Omega^{y_2}) \subset \partial\phi_{y_2}(\Omega^{y_2}) \,. \tag{26b}$$

Proof. The strategy of proof closely follows [4, pp. 10–11]. We write out the argument for (26a) explicitly; the proof of (26b) is analogous.

Fix $(p_2, p_3) \in \partial\psi_{y_1}(\Omega^{y_1})$. Then there is $(y_2, z^0) \in \Omega^{y_1}$ so that $(p_2, p_3) \in \partial\psi_{y_1}(y_2, z^0)$ and therefore

$$\psi_{y_1}(x_2, z) \geq \psi_{y_1}(y_2, z^0) + p_2\,(x_2 - y_2) + p_3\,(z - z^0) \tag{27}$$

for all $(x_2, z) \in \Omega^{y_1}$. Subtracting $\phi_{y_1}(x_2, z)$ from (27), taking the supremum on the right hand side, and using that $\psi_{y_1} \geq \phi_{y_1}$, we find that

$$a \equiv \sup_{(x_2,z)\in\overline{\Omega}^{y_1}} \bigl(\psi_{y_1}(y_2, z^0) + p_2\,(x_2 - y_2) + p_3\,(z - z^0) - \phi_{y_1}(x_2, z)\bigr) \geq 0 \,. \tag{28}$$

Since Ω and Ω^{y_1} are bounded and ϕ_{y_1} is continuous, the supremum in (28) is attained at some $(x_2^1, z^1) \in \overline{\Omega}^{y_1}$, so that, by the definition of a,

$$\begin{aligned} \phi_{y_1}(x_2, z) &\geq \psi_{y_1}(y_2, z^0) + p_2\,(x_2 - y_2) + p_3\,(z - z^0) - a \\ &= \phi_{y_1}(x_2^1, z^1) + p_2\,(x_2 - x_2^1) + p_3\,(z - z^1) \end{aligned} \tag{29}$$

for all $(x_2, z) \in \Omega^{y_1}$. Clearly, the right hand side of (29) defines a supporting hyperplane for ϕ_{y_1} at (x_2^1, z^1). When $(x_2^1, z^1) \in \Omega^{y_1}$, we conclude that $(p_2, p_3) \in \partial\phi_{y_1}(\Omega^{y_1})$ and we are done. Otherwise, when $(x_2^1, z^1) \in \partial\Omega^{y_1}$, then, by assumption, $\psi_{y_1}(x_2^1, z^1) = \phi_{y_1}(x_2^1, z^1)$. Further, by continuity of ψ, we may let $(x_2, z) \to (x_2^1, z^1)$ in (27), so that

$$\phi_{y_1}(x_2^1, z^1) \geq \psi_{y_1}(y_2, z^0) + p_2\,(x_2^1 - y_2) + p_3\,(z^1 - z^0) = \phi_{y_1}(x_2^1, z^1) + a \,. \tag{30}$$

Since $a \geq 0$, this implies $a = 0$. Therefore, $\psi_{y_1}(y_2, z^0) \leq \phi_{y_1}(y_2, z^0)$. By assumption, the reverse inequality is also true, so that $\psi_{y_1}(y_2, z^0) = \phi_{y_1}(y_2, z^0)$. With these provisions, the first line in (29) reads

$$\phi_{y_1}(x_2, z) \geq \phi_{y_1}(y_2, z^0) + p_2\,(x_2 - y_2) + p_3\,(z - z^0) \,. \tag{31}$$

Clearly, the right hand side defines a supporting hyperplane for ϕ_{y_1} at (y_2, z^0), so that $(p_2, p_3) \in \partial\phi_{y_1}(\Omega^{y_1})$ in this case, too. □

Lemma 5. *Let $\Omega \subset \mathbb{R}^3$ be open and bounded, and let $\phi, \psi \in C(\overline{\Omega})$ be bi-planar convex. If $\phi \leq \psi$ in Ω and $\phi = \psi$ on $\partial\Omega$, then*

$$\mu_\psi(\Omega) \leq \mu_\phi(\Omega) \,. \tag{32}$$

Proof. Lemma 4 implies, for any fixed x_1, the inclusion $\partial\psi_{x_1}(\Omega^{x_1}) \subset \partial\phi_{x_1}(\Omega^{x_1})$. Hence,

$$M_{23}\psi_{x_1}(\Omega^{x_1}) \leq M_{23}\phi_{x_1}(\Omega^{x_1})\,. \tag{33}$$

Integrating this inequality with respect to x_1, we obtain $\mu_1^\psi(\Omega) \leq \mu_1^\phi(\Omega)$, where μ_1^ϕ and μ_1^ψ are the measures μ_1 corresponding to ϕ and ψ, respectively.

An analogous argument yields $\mu_2^\psi(\Omega) \leq \mu_2^\phi(\Omega)$, where μ_2^ϕ and μ_2^ψ are the measures μ_2 corresponding to ϕ and ψ respectively. Since $\mu_\phi = \mu_1^\phi + \mu_2^\phi$, the proof is complete. □

Finally, superadditivity of the Monge–Ampère measure (e.g. [4, p. 17]) directly implies the following inequality.

Lemma 6. *Let $\Omega \subset \mathbb{R}^3$ be open and bounded and let $\phi, \psi \in C(\overline{\Omega})$ be bi-planar convex functions. Then*

$$\mu_{\phi+\psi}(E) \geq \mu_\phi(E) + \mu_\psi(E) \tag{34}$$

for any Borel set $E \subset \Omega$.

2.3. **Comparison principle.** The first central result which carries over from the classical Monge–Ampère measure to the bi-planar case is the comparison principle. The proof is a close adaptation of [4, pp. 16–17].

Theorem 7 (Comparison principle)**.** *Let $\Omega \subset \mathbb{R}^3$ be open and bounded and let $\phi, \psi \in C(\overline{\Omega})$ be bi-planar convex functions such that*

$$\mu_\phi \leq \mu_\psi \quad \textit{on } \Omega\,. \tag{35}$$

Then

$$\min_{x\in\overline{\Omega}}(\phi - \psi)(x) = \min_{x\in\partial\Omega}(\phi - \psi)(x)\,. \tag{36}$$

Proof. Suppose (36) does not hold, i.e.,

$$a \equiv \min_{x\in\overline{\Omega}}(\phi - \psi)(x) < \min_{x\in\partial\Omega}(\phi - \psi)(x) \equiv b\,. \tag{37}$$

Choose $x^0 = (x_1^0, x_2^0, z^0) \in \Omega$ such that $a = \phi(x^0) - \psi(x^0)$. Since Ω is bounded, there exists $\delta > 0$ such that $\delta\,(\operatorname{diam}\Omega)^2 < (b-a)/2$. Set

$$\varphi(x) = \psi(x) + \delta\,|x - x^0|^2 + \frac{b+a}{2} \tag{38}$$

and consider the set

$$G = \{x = (x_1, x_2, z) \in \overline{\Omega} \colon \phi(x) < \varphi(x)\}\,. \tag{39}$$

On the one hand, $x^0 \in G$. Indeed, using (38) and $b - a > 0$, we find that

$$\varphi(x^0) = \psi(x^0) + \frac{b+a}{2} = \phi(x^0) + \frac{b-a}{2} > \phi(x^0)\,. \tag{40}$$

On the other hand, for $x \in \partial\Omega$, we have $\phi(x) - \psi(x) \geq b$ so that, by (38),

$$\phi(x) - \varphi(x) \geq b - \frac{b+a}{2} - \delta\,|x - x^0|^2 > \frac{b-a}{2} - \delta\,(\operatorname{diam}\Omega)^2 > 0\,. \tag{41}$$

Hence, $G \cap \partial\Omega = \emptyset$ and consequently $\partial G = \{x \in \Omega \colon \varphi(x) = \phi(x)\}$. Hence, using Lemma 5 and Lemma 6, we conclude

$$\mu_\phi(G) \geq \mu_\varphi(G) \geq \mu_\psi(G) + 8\,\delta^2\,|G|\,, \tag{42}$$

which contradicts (35). This completes the proof. □

2.4. **Generalized solutions.** The bi-planar Monge–Ampère measure can be used to define the analog of Aleksandrov generalized solutions for the bi-planar Monge–Ampère equation.

Definition 8. Let $\Omega \subset \mathbb{R}^3$ be open and let ν be a Borel measure on Ω. Then the bi-planar convex function $\phi \in C(\Omega)$ is a generalized solution of the bi-planar Monge–Ampère equation

$$\det D^2_{x_1 z}\phi + \det D^2_{x_2 z}\phi = \nu \tag{43}$$

if the bi-planar Monge–Ampère measure associated with ϕ equals ν.

The following statement is then a direct consequence of Lemma 3.

Proposition 9. *Let $\Omega \subset \mathbb{R}^3$ be open, $\phi \in C^2(\overline{\Omega})$, and suppose the Borel measure ν is absolutely continuous with respect to the Lebesgue measure with non-negative density function $f \in C(\Omega)$. Then ϕ is a generalized solution of* (43) *if and only if*

$$\det D^2_{x_1 z}\phi + \det D^2_{x_2 z}\phi = f \quad \text{in } \Omega\,. \tag{44}$$

Finally, the comparison principle implies uniqueness of generalized solutions.

Theorem 10. *Let $\Omega \subset \mathbb{R}^3$ be open and bounded, ν a Borel measure on Ω, and $g \in C(\partial\Omega)$. If ϕ_1 and ϕ_2 are generalized solutions to the Dirichlet problem*

$$\begin{cases} \det D^2_{x_1 z}\phi + \det D^2_{x_2 z}\phi = \nu & \text{in } \Omega \\ \phi = g & \text{on } \partial\Omega\,, \end{cases} \tag{45}$$

then $u_1 = u_2$.

3. Non-existence results

In this section, we present main results of the paper with corresponding examples: there is a domain and boundary data such that no generalized solution to the Dirichlet problem with zero or constant right hand side exist.

3.1. **Non-existence of convex generalized solutions.** We begin the discussion with a weaker result, namely, there is no generalized solution in the class of convex functions. This construction illustrates in a particularly transparent way how convexity over-constrains the system. We begin with a simple observation.

Lemma 11. *Let A be a symmetric positive semi-definite matrix, written as*

$$A = \begin{pmatrix} B & C \\ C^T & S \end{pmatrix}. \tag{46}$$

Then $\det S = 0$ *implies* $\det A = 0$.

Proof. By assumption, the submatrix S must also be symmetric positive semi-definite. Moreover, since S is singular, we can take a nonzero $v \in \operatorname{Ker} S$ and set $w^T = (0, v^T)$. Then $w^T A w = v^T S v = 0$, so A cannot be strictly positive definite. This implies $\det A = 0$. □

Lemma 11 implies that if problem (45) with $\nu = 0$ has a convex solution, then the solution also satisfies the classical homogeneous Monge–Ampère equation

$$\begin{cases} \det D^2\phi = 0 & \text{in } \Omega\,, \\ \phi = g & \text{on } \partial\Omega\,. \end{cases} \tag{47}$$

From this observation, we conclude the following.

Proposition 12. *There exist a bounded domain Ω and a continuous function g defined on $\partial\Omega$ such that problem (45) with $\nu = 0$ has no solution in the class of convex functions.*

Proof. Let $U \subset \mathbb{R}^2$ be a bounded domain and $\Omega \equiv U \times (0,1)$. Then Ω is a bounded domain in $\mathbb{R}^3$. Let g be a restriction of the function $\phi(x) = x_1^2 + z^2$ to the set $\partial\Omega$. Then ϕ is a solution of (47). This solution is unique [4, 9]. Now suppose ϕ_1 is a convex solution of (45) with the same boundary data g. By Lemma 11, ϕ_1 is also a solution of (47), i.e., $\phi_1(x) = \phi(x) = x_1^2 + z^2$. But

$$\det D^2_{x_1 z}\phi + \det D^2_{x_2 z}\phi = 4 \neq 0 \,. \tag{48}$$

Thus, the problem (45) with boundary data g does not have a convex solution. □

Remark 13. The existence of generalized solution for the homogeneous Dirichlet problem (47) for strictly convex domains is well known, e.g. [4, Theorem 1.5.2]. Clearly, the difference between convexity and strict convexity of the domain is not an issue here. A similar counter-example can be produced, for example, on the unit ball in $\mathbb{R}^3$: Let g be the restriction of the function $\phi(x) = x_1^2 + z^2$ to the unit sphere $\partial\Omega$. Then, proceeding as in the proof of Proposition 12, problem (45) with boundary data g does not have a convex solution.

Remark 14. Actually, g in the proof of Proposition 12 can be written as $g(x) = 1 - x_2^2$. Then $\phi_1(x) = 1 - x_2^2$ is the unique concave solution to the problem (45) and also a concave solution to problem (47), unique by the concave analog to Theorem 10.

Remark 15. For boundary data $g(x) = x_1^2 - z^2$ on the unit sphere Ω, there is neither a concave or a convex solution to problem (45). Indeed, since $x_1^2 + x_2^2 + z^2 = 1$, $g(x)$ can be written as

$$g(x)\big|_{\partial\Omega} = x_1^2 - (1 - x_1^2 - x_2^2) = 2x_1^2 + x_2^2 - 1 \,. \tag{49}$$

Hence, $\phi(x) = 2x_1^2 + x_2^2 - 1$ is the unique convex solution to the classical Monge–Ampère equation (47). On the other hand,

$$g(x)\big|_{\partial\Omega} = (1 - x_2^2 - z^2) - z^2 = 1 - x_2^2 - 2z^2 \,. \tag{50}$$

The function $\phi_1(x) = 1 - x_2^2 - 2z^2$ is the concave solution to the problem (47). However, ϕ_1 does not satisfy problem (45).

3.2. **Non-existence of bi-planar convex generalized solutions.** We now refine the construction to show that relaxing the constraint from convexity to bi-planar convexity does not help: there exists a smooth boundary condition for the Dirichlet problem such that the bi-planar Monge–Ampère equation does not have a generalized solution.

Theorem 16. *Let Ω be the unit ball centered at the origin and $\lambda \in [0,8)$ be fixed. Then the problem*

$$\begin{cases} \det D^2_{x_1 z}\phi + \det D^2_{x_2 z}\phi = \lambda & \text{in } \Omega \\ \phi(x) = x_1^2 - x_2^2 & \text{on } \partial\Omega \end{cases} \tag{51}$$

has no generalized solution in the class of continuous bi-planar convex functions.

Proof. Suppose that problem (51) has a generalized solution $\phi \in C(\overline{\Omega})$. Bi-planar convexity implies that each of the measures μ_1 and μ_2 is positive. As $\mu_1 + \mu_2 = \lambda$, we have

$$0 \le \mu_i \le \lambda \tag{52}$$

in the sense of measure for $i = 1, 2$.

Now consider two classical Monge–Ampère equations. First, fix $|x_2| < 1$ and consider

$$\begin{cases} \det D^2_{x_1 z}\phi_1 = \lambda & \text{in } \Omega^{x_2}\,, \\ \phi_1 = x_1^2 - x_2^2 & \text{on } \partial\Omega^{x_2}\,. \end{cases} \tag{53}$$

The smooth function

$$\phi_1(x) = x_1^2 - x_2^2 + \frac{\sqrt{1+\lambda}-1}{2}\,(x_1^2 + x_2^2 + z^2 - 1) \tag{54}$$

solves (53). Further, ϕ_1 is convex, therefore it is the unique generalized solution; see, e.g., [4, Corollary 1.4.7]. Similarly, we fix $|x_1| < 1$ and consider the problem

$$\begin{cases} \det D^2_{x_2 z}\phi_2 = \lambda & \text{in } \Omega^{x_1}\,, \\ \phi_2 = x_1^2 - x_2^2 & \text{on } \partial\Omega^{x_1}\,. \end{cases} \tag{55}$$

Here, the unique generalized solution is the smooth convex function

$$\phi_2(x) = 2x_1^2 + z^2 - 1 + \frac{\sqrt{1+\lambda}-1}{2}\,(x_1^2 + x_2^2 + z^2 - 1)\,. \tag{56}$$

We further set

$$\psi_1(x) = x_1^2 - x_2^2 \quad \text{and} \quad \psi_2(x) = 2x_1^2 + z^2 - 1\,. \tag{57}$$

By direct computation, $\det D^2_{x_i z}\phi_i = \lambda$ and $\det D^2_{x_i z}\psi_i = 0$ for $i = 1, 2$. Applying comparison principle, Theorem 7, to each of the inequalities in (52), we conclude that, for every $x \in \Omega$,

$$\phi_i(x) \le \phi(x) \le \psi_i(x)\,. \tag{58}$$

In particular, for $x = (0, 0, 0)$, we get

$$\frac{1-\sqrt{1+\lambda}}{2} \le \phi(0,0,0) \le 0 \tag{59}$$

and

$$\frac{1-\sqrt{1+\lambda}}{2} - 1 \le \phi(0,0,0) \le -1\,. \tag{60}$$

For $0 \le \lambda < 8$, we find $-1 < \frac{1-\sqrt{1+\lambda}}{2} \le \phi(0,0,0) \le -1$, a contradiction. Hence, (51) cannot have a generalized solution. $\square$

Remark 17. Note that problem (51) has a classical solution,

$$\phi(x) = x_1^2 - x_2^2 + \sqrt{\frac{\lambda}{8}}\,(x_1^2 + x_2^2 + z^2 - 1) \tag{61}$$

for any $\lambda \ge 0$. When $\lambda \ge 8$, this function is convex, hence bi-planar convex. When $0 \le \lambda < 8$, then $D^2_{x_2 z}\phi$ is not positive semi-definite so that ϕ is not bi-planar convex. However, even in this case, problem (51) satisfies the ellipticity condition (5) for the bi-planar Monge–Ampère equation.

4. Discussion

Our counter-examples show that ellipticity for the bi-planar Monge–Ampère equation is a substantially weaker condition than bi-planar convexity, while bi-planar convexity implies ellipticity. In contrast, for the classical Monge–Ampère equation ellipticity and convexity coincide, which makes Aleksandrov generalized solutions a useful concept. For the bi-planar equation, there is a "gap" between ellipticity and bi-planar convexity, so that the requirements of convex analysis, necessary to obtain generalized solutions in the sense of Aleksandrov, over-constrain the system. Therefore, a useful solution concept for the bi-planar Monge–Ampère equation requires a different setting, possibly in more traditional function space setting as, for example, in [6, 7]. This question is left open for future work.

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References

[1] A. D. Aleksandrov: *Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it* (Russian), Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 6, 1939, 3–35.

[2] J. S. Allen, D. D. Holm, and P. A. Newberger: *Toward an extended-geostrophic Euler–Poincaré model for mesoscale oceanographic flow*, in "Large-Scale Atmosphere-Ocean Dynamics 1: Analytical Methods and Numerical Models," J. Norbury & I. Roulstone, eds., Cambridge University Press, Cambridge, 2002, 101–125.

[3] L. Caffarelli, X. Cabré: *Fully nonlinear elliptic equations*, Amer. Math. Soc., Colloquium Publications 43, 1995.

[4] C. E. Gutiérrez: *The Monge–Ampère equation*, Birkhäuser, Boston, 2001.

[5] Q. Han: *Nonlinear elliptic equations of the second order*, Amer. Math. Soc., Graduate Studies in Mathematics 171, 2016.

[6] P.-L. Lions: *Sur les équations de Monge–Ampère. I.*, Manuscripta Math. 41, 1983, 1–43.

[7] P.-L. Lions: *Sur les équations de Monge–Ampère*, Arch. Rational Mech. Anal. 89, 1985, 93–122.

[8] M. Oliver and S. Vasylkevych: *Generalized large-scale semigeostrophic approximations for the f-plane primitive equations*, J. Phys. A: Math. Theor. 49, 2016, 184001.

[9] W. Ożański: *A generalised comparison principle for the Monge–Ampère equation and the pressure in 2D fluid flows*, C. R. Math. Acad. Sci. Paris 356(2), 2018, 198–206.

[10] R. Salmon: *Large-scale semi-geostrophic equations for use in ocean circulation models*, J. Fluid Mech. 318, 1996, 85–105.

[11] N. S. Trudinger and X.-J. Wang: *The Monge–Ampère equation and its geometric applications*, Handbook of Geometric Analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, 467–524.

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